

ZARISKI DENSITY OF CRYSTALLINE REPRESENTATIONS FOR ANY p -ADIC FIELD.

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ABSTRACT. The aim of this article is to prove Zariski density of crystalline representations in the rigid analytic space associated to the universal deformation ring of a d -dimensional mod p representation of $\mathrm{Gal}(\overline{K}/K)$ for any d and for any p -adic field K . This is a generalization of the results of Colmez, Kisin ($d = 2$, $K = \mathbb{Q}_p$), of the author ($d = 2$, any K), of Chenevier (any d , $K = \mathbb{Q}_p$). A key ingredient for the proof is to construct a p -adic family of trianguline representations. In this article, we construct (an approximation of) this family by generalizing Kisin's theory of finite slope subspace X_{fs} for any d and for any K .

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1. INTRODUCTION.

1.1. Background. Let K be a finite extension of \mathbb{Q}_p and $d \in \mathbb{Z}_{\geq 1}$. Let E be a sufficient large finite extension of K and \mathcal{O} be the integer ring of E and \mathbb{F} be the residue field of E . Let \overline{V} be a \mathbb{F} -representation of $G_K := \mathrm{Gal}(\overline{K}/K)$ of rank d , i.e. d -dimensional \mathbb{F} -vector space with a continuous \mathbb{F} -linear G_K -action. Let $\mathcal{C}_{\mathcal{O}}$ be the category of Artin local \mathcal{O} -algebra with the residue field \mathbb{F} . We consider a deformation functor $D_{\overline{V}} : \mathcal{C}_{\mathcal{O}} \rightarrow (\mathrm{Sets})$ defined by $D_{\overline{V}}(A) := \{\text{equivalent classes of deformations of } \overline{V} \text{ over } A\}$. We assume that $\mathrm{End}_{\mathbb{F}[G_K]}(\overline{V}) = \mathbb{F}$, then $D_{\overline{V}}$ is represented by the universal deformation ring $R_{\overline{V}}$. Let $\mathfrak{X}_{\overline{V}}$ be the rigid analytic space associated to $R_{\overline{V}}$, then the points of $\mathfrak{X}_{\overline{V}}$ correspond to p -adic representations of G_K with a mod p reduction isomorphic to \overline{V} . We define the subset $\mathfrak{X}_{\overline{V}, \mathrm{reg-cris}}$

2010 Mathematical Subject Classification 11F80 (primary), 11F85 (secondary). Keywords: p -adic Hodge theory, trianguline representations, B -pairs.

of $\mathfrak{X}_{\overline{V}}$ by

$$\mathfrak{X}_{\overline{V}, \text{reg-cris}} := \{[V] \in \mathfrak{X}_{\overline{V}} \mid V \text{ is crystalline with distinct Hodge-Tate weights} \}.$$

We denote by $\overline{\mathfrak{X}}_{\overline{V}, \text{reg-cris}}$ the Zariski closure of $\mathfrak{X}_{\overline{V}, \text{reg-cris}}$ in $\mathfrak{X}_{\overline{V}}$. A main theorem of this article is following (see theorem 4.5).

Theorem 1.1. *If \overline{V} is absolutely irreducible and satisfies $\zeta_p \in K$ or $\overline{V} \not\cong \overline{V}(\omega)$, then we have an equality*

$$\overline{\mathfrak{X}}_{\overline{V}, \text{reg-cris}} = \mathfrak{X}_{\overline{V}},$$

where ω is the mod p cyclotomic character.

This theorem is a generalization (for d :general, K :general) of the results of Colmez, Kisin ([Co08], [Ki10], $d = 2$, $K = \mathbb{Q}_p$) and the author ([Na10], $d = 2$, K :general) and Chenevier (theorem A of [Ch10], d :general, $K = \mathbb{Q}_p$). When $d = 2$ and $K = \mathbb{Q}_p$, the results of Colmez and Kisin played many crucial roles in the study of p -adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$.

The idea of the proof is the same as those of [Co08], [Ki10], [Na10], [Ch10], i.e. we re-interpret purely locally the argument of infinite fern of Gouvêa-Mazur by using the concept of trianguline representations. Inspired by a Kisin's work ([Ki03]) on a p -adic Hodge theoretic study of Coleman-Mazur eigencurve (where he proved that two dimensional p -adic representations of $G_{\mathbb{Q}_p}$ parametrized by Coleman-Mazur eigencurve are trianguline), Colmez ([Co08]) defined and studied trianguline representations (for $K = \mathbb{Q}_p$) by using the theory of (φ, Γ) -modules over the Robba ring. In [Na09], the author of this article generalized his results, studied trianguline representations for any K by using the theory of B -pair which was defined by Berger [Be08].

For the proof of the main theorem, there are two key ingredients, one is deformation theory of trianguline representations and the other is to construct a p -adic family of trianguline representations in which the subset consisting of all the crystalline points is Zariski dense. For deformation theory of trianguline representations, we have already obtained satisfying results in [BeCh09], [Ch09b] (when $K = \mathbb{Q}_p$) and in [Na10] (for general K).

A more important ingredient (and a main theme of this article) is to construct a p -adic family of trianguline representations, which can be seen as a p -adic avatar of Gouvêa-Mazur's infinite fern consisting of overconvergent modular forms. When $K = \mathbb{Q}_p$ and $d = 2$, two different constructions by Colmez and Kisin are known. Colmez ([Co08]) explicitly constructed (more generally) a p -adic family of rank two trianguline (φ, Γ) -modules over the Robba ring of affinoid. On the other hands, Kisin ([Ki03]) constructed X_{f_s} (which is a Zariski closed subspace of $\mathfrak{X}_{\overline{V}} \times_E \mathbb{G}_{m/E}^{\text{an}}$) an approximation of a p -adic family of trianguline representations, which we call finite slope subspace. When $K = \mathbb{Q}_p$, for any $d \in \mathbb{Z}_{\geq 1}$, Chenevier ([Ch10]) recently generalized Colmez's construction, he constructed a universal p -adic family of rank d trianguline (φ, Γ) -modules by establishing the cohomology

theory of (φ, Γ) -modules over the Robba ring of affinoid. Because his construction depends on the explicit structure of (φ, Γ) -modules only for $K = \mathbb{Q}_p$, we cannot directly generalize his results for any K -case. A main feature of this article is to modify and to generalize Kisin's X_{fs} for any K and for any d (when $d = 2$, we have already done this in [Na10]) of which we explain below more precisely.

1.2. Overview. Here, we first recall briefly the definition of trianguline representations. First, the category of E -representation of G_K can be naturally embedded in the category of E - B -pairs of G_K . For an E -representation V , we denote by $W(V)$ the associated E - B -pair. We say that V is a split trianguline E -representation if $W(V)$ can be written as a successive extension of rank one E - B -pairs, i.e. there exists a filtration $\mathcal{T} : 0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_d = W(V)$ (we call \mathcal{T} a triangulation of V) by E - B -pairs W_i such that W_i/W_{i-1} are rank one E - B -pairs for any i . Rank one E - B -pairs can be classified by the set of continuous homomorphisms $\delta : K^\times \rightarrow E^\times$ ([Co08], [Na09]). For a continuous homomorphism $\delta : K^\times \rightarrow E^\times$, we denote by $W(\delta)$ the rank one E - B -pair defined by δ . By the definition of \mathcal{T} , there exists a set $\{\delta_i\}_{i=1}^d$ ($\delta_i : K^\times \rightarrow E^\times$) such that $W_i/W_{i-1} \xrightarrow{\sim} W(\delta_i)$ for any i , which we call the parameter of \mathcal{T} . Therefore, to construct a p -adic family of trianguline representations, we first need to construct a universal p -adic family of continuous homomorphisms $\delta : K^\times \rightarrow E^\times$, which we now recall the definition. Let \mathcal{T} and \mathcal{W} be the rigid analytic spaces over E which represent the functor $\mathcal{T}(A) := \{\delta : K^\times \rightarrow A^\times \text{ continuous homomorphism}\}$ and $\mathcal{W}(A) := \{\delta : \mathcal{O}_K^\times \rightarrow A^\times \text{ continuous homomorphism}\}$ (for any affinoid A) respectively. If we fix a uniformizer π_K of K , we have an isomorphism $\mathcal{T} \xrightarrow{\sim} \mathcal{W} \times_E \mathbb{G}_{m/E}^{\text{an}} : \delta \mapsto (\delta|_{\mathcal{O}_K^\times}, \delta(\pi_K))$. In [Na10], we modified and generalized Kisin's X_{fs} for any K , i.e. twisting by a universal character, we constructed X_{fs} as a Zariski closed subspace of $\mathfrak{X}_{\overline{V}} \times_E \mathcal{T}$ instead of $\mathfrak{X}_{\overline{V}} \times_E \mathbb{G}_{m/E}^{\text{an}}$. In this article, we generalize the construction of [Na10] for any d , we construct X_{fs} (which we denote by $\mathcal{E}_{\overline{V}}$) as a Zariski closed subspace of $Z := \mathfrak{X}_{\overline{V}} \times_E \mathcal{T}^{\times(d-1)}$. Let V' be a split trianguline E' -representation with a triangulation \mathcal{T} whose parameter is $\{\delta_i\}_{i=1}^d$ such that $[V] \in \mathfrak{X}_{\overline{V}}$ for a finite extension E' of E . From the pair (V, \mathcal{T}) , we define an E' -rational point $z_{(V, \mathcal{T})} := ([V], \delta_1, \delta_2, \dots, \delta_{d-1}) \in Z(E')$. The key main theorem of this article is the following (for more precise statements, see corollary 3.4, proposition 3.6, theorem 3.9)

Theorem 1.2. *There exists a Zariski closed subspace $\mathcal{E}_{\overline{V}}$ of Z satisfying the following (1) and (2).*

- (1) *For any point $z = ([V], \delta_1, \dots, \delta_{d-1}) \in \mathcal{E}_{\overline{V}}(E')$ and for any embedding $\sigma : K \hookrightarrow E$, the σ -part of Hodge-Tate polynomial of V is equal to $\prod_{i=1}^d (T - \frac{\partial \delta_i(x)}{\partial \sigma(x)}|_{x=1}) \in E'[T]$, where $\delta_d := (\det(V) \circ \text{rec}_K) / \prod_{i=1}^{d-1} \delta_i$ ($\text{rec}_K : K^\times \hookrightarrow G_K^{\text{ab}}$ is the reciprocity map).*
- (2) *If (V, \mathcal{T}) as above satisfies the following conditions (i) and (ii) and (iii),*

- (i) $\text{End}_{E'[G_K]}(V) = E'$,
- (ii) For any $i < j$, $\delta_j/\delta_i \neq \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\leq 0}$,
- (iii) For any $i < j$, $\delta_i/\delta_j \neq |N_{K/\mathbb{Q}_p}|_p \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\geq 1}$,

then the point $z_{(V, \mathcal{T})} \in Z$ defined above is contained in $\mathcal{E}_{\overline{V}}$.

Moreover, if V satisfies one of the following additional conditions (iv), (v),

- (iv) V is potentially crystalline and

$$\{a \in D_{\text{cris}}((\wedge^i V)(\prod_{j=1}^i \tilde{\delta}_j^{-1})) \mid \exists n \geq 1 \text{ such that } (\varphi^f - \prod_{j=1}^i \delta_j(\pi_K))^n a = 0\}$$

is a free of rank one $K_0 \otimes_{\mathbb{Q}_p} E$ -module for any $1 \leq i \leq d-1$,

- (v) For any $1 \leq i \leq d-1$ and for any $1 \leq l_1 < l_2 < \dots < l_i \leq d$ such that $(l_1, \dots, l_i) \neq (1, 2, \dots, i)$, we have $(\sum_{j=1}^i k_{l_j, \sigma} - \sum_{j=1}^i k_{j, \sigma}) \notin \mathbb{Z}_{\leq 0}$ for any $\sigma \in \mathcal{P}$, where we define $k_{i, \sigma} := \frac{\partial \delta_i(x)}{\partial \sigma(x)}|_{x=1} \in E'$,

then $\mathcal{E}_{\overline{V}}$ is smooth at $z_{(V, \mathcal{T})}$ of dimension $[K : \mathbb{Q}_p] \frac{d(d+1)}{2} + 1$.

The idea for the construction of $\mathcal{E}_{\overline{V}}$ is to generalize Kisin's construction by using a technique of exterior product ([BeCh09], see proposition 3.10) and using the results of [Na09] and [Na10]. If V is a split trianguline E representation as above. Let $\tilde{\delta}_i : G_K^{\text{ab}} \rightarrow E^\times$ be the continuous character defined by $\tilde{\delta}_i \circ \text{rec}_K|_{\mathcal{O}_K^\times} = \delta_i|_{\mathcal{O}_K^\times}$ and $\tilde{\delta}_i(\text{rec}_K(\pi_K)) := 1$ for any i . Then, we can show that $(*)$ " $D_{\text{cris}}^+(\wedge^i V)(\prod_{j=1}^i \tilde{\delta}_j^{-1})^{\varphi^f = \prod_{j=1}^i \delta_j(\pi_K)}$ are non-zero for any $1 \leq i \leq d-1$ ". We construct $\mathcal{E}_{\overline{V}}$ as a subspace of Z essentially parametrizing the points $z = (V, \delta_1, \dots, \delta_{d-1})$ with this property $(*)$ (and the property (1) of theorem 1.2).

Concerning to Zariski density of crystalline points in $\mathcal{E}_{\overline{V}}$, we prove the following theorem (theorem 3.15).

Theorem 1.3. *Let (V, \mathcal{T}) be a pair satisfying all the conditions of theorem 1.2 and let U be an admissible open neighborhood of $z := z_{(V, \mathcal{T})}$ in $\mathcal{E}_{\overline{V}}$, then there exists an admissible open neighborhood U' of z in U in which the subset consisting of the points $z' = ([V'], \delta'_1, \dots, \delta'_{d-1}) \in U'$ such that V' is crystalline with distinct Hodge-Tate weights are Zariski dense in U' .*

The main theorem 1.1 follows from these two theorems 1.2, 1.3 and from the deformation theory of trianguline (in particular, generic crystalline or benign) representations developed in [Ch09b] and [Na10]. For the proof of theorem 1.3, we need to prove that, oppositely, if a point $z = ([V], \delta_1, \dots, \delta_{d-1}) \in \mathcal{E}_{\overline{V}}$ satisfying the condition $(*)$ above (and some conditions on $\{\delta_i\}_{i=1}^d$) then V is a split trianguline and crystalline. This problem was pointed out to the author by Chenevier. Concerning to this problem, we prove some propositions (see proposition 3.13, proposition 3.14) by using slope arguments, which the author thinks important in the study of eigenvarieties.

Finally, we remark that we can regard this paper as the case of the pair $(\mathrm{GL}_n, B_n, T_n)$ where B_n is the subgroup of upper triangular matrices and T_n is the subgroup of diagonal matrices, the author thinks that to generalize the main theorem for the pair (G, B, T) (where G is a reductive group and B is a Borel subgroup of G and T is a maximal torus contained in B) is also an interesting problem, which the author will study in future works.

Acknowledgement. A part of this article was written during a stay at École Polytechnique. The author would like to thank Gaëtan Chenevier for many valuable discussions and for his kind hospitality and thanks also all the people there for their kind hospitality. The author would like to thank Kenichi Bannai for constantly encouraging the author and thank Go Yamashita and Seidai Yasuda for many valuable discussions and pointing out some mistakes in the previous version of this article. The author would like to thank all the people at Keio University related to the JSPS ITP program which enabled my stay. The author is also supported by KAKENHI (21674001).

Notation. Let p be a prime number. K be a finite extension of \mathbb{Q}_p , \overline{K} be the algebraic closure of K , K_0 the maximal unramified extension of \mathbb{Q}_p in K , K^{nor} the Galois closure of K in \overline{K} . Let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K equipped with pro-finite topology. \mathcal{O}_K is the ring of integers of K , $\pi_K \in \mathcal{O}_K$ is a fixed uniformizer of K , $k := \mathcal{O}_K/\pi_K \mathcal{O}_K$ is the residue field of K , $q = p^f := \#k$ is the order of k , $\chi_p : G_K \rightarrow \mathbb{Z}_p^\times$ is the p -adic cyclotomic character (i.e. $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$ for any p^n -th roots of unity and for any $g \in G_K$). Let $\mathbb{C}_p := \widehat{\overline{K}}$ be the p -adic completion of \overline{K} and $\mathcal{O}_{\mathbb{C}_p}$ be its ring of integers. We denote by v_p the normalized valuation on \mathbb{C}_p^\times such that $v_p(p) = 1$. Let $|\cdot|_p : E \rightarrow \mathbb{Q}_{\geq 0}$ be the norm such that $|p|_p := \frac{1}{p}$. Let $N_{K/\mathbb{Q}_p} : K^\times \rightarrow \mathbb{Q}_p^\times$ be the norm and we define by $|N_{K/\mathbb{Q}_p}|_p := |\cdot|_p \circ N_{K/\mathbb{Q}_p}$. Let E be a finite extension of \mathbb{Q}_p in \overline{K} such that $K^{\mathrm{nor}} \subseteq E$. In this paper, we use the notation E as a coefficient field of representations. We denote by $\mathcal{P} := \{\sigma : K \hookrightarrow \overline{K}\} = \{\sigma : K \hookrightarrow E\}$ the set of \mathbb{Q}_p -algebra homomorphisms from K to \overline{K} (or E). Let $\chi_{\mathrm{LT}} : G_K \rightarrow \mathcal{O}_K^\times \hookrightarrow \mathcal{O}_E^\times$ be the Lubin-Tate character associated with the fixed uniformizer π_K . Let $\mathrm{rec}_K : K^\times \rightarrow G_K^{\mathrm{ab}}$ be the reciprocity map of local class field theory such that $\mathrm{rec}_K(\pi_K)$ is a lifting of the inverse of q -th power Frobenius on k , then $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K : K^\times \rightarrow \mathcal{O}_K^\times$ satisfies $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K(\pi_K) = 1$ and $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K|_{\mathcal{O}_K^\times} = \mathrm{id}_{\mathcal{O}_K^\times}$. For any topological ring A (for example $A = E, \mathcal{O}, \mathbb{F}$), we say that V_A is an A -representation (of G_K) if V_A is a finite free A -module with a continuous A -linear G_K -action.

2. REVIEW OF B -PAIRS AND TRIANGULINE REPRESENTATIONS

In this section, we recall the definition of B -pairs and trianguline representations and then recall some of their fundamental properties which we will use in later sections (see [Be08] or [Na09], [Na10] for more details).

Let B_{cris} and B_{dR}^+ and B_{dR} be Fontaine's p -adic period rings ([Fo94]). Let $B_e := B_{\text{cris}}^{\varphi=1}$ the φ -fixed part of B_{cris} . These rings are equipped with continuous semi-linear G_K -actions. Let $t = \log[\varepsilon] \in B_{\text{cris}}^{\varphi=p} \cap \text{Fil}^1 B_{\text{dR}}^+$ be a period of the inverse of the p -adic cyclotomic character χ_p . Let \mathcal{C}_E be the category of Artin local E -algebras A such that $A/\mathfrak{m}_A \xrightarrow{\sim} \mathbb{F}$ where \mathfrak{m}_A is the maximal ideal of A .

For any $A \in \mathcal{C}_E$, we recall the definition of A - B -pair which is the A -coefficient version of B -pair (see definition 2.10 and lemma 2.11 of [Na10]).

Definition 2.1. We say a pair $W := (W_e, W_{\text{dR}}^+)$ an A - B -pair (of G_K) if

- (1) W_e is a finite free $B_e \otimes_{\mathbb{Q}_p} A$ -module with a continuous semi-linear G_K -action.
- (2) W_{dR}^+ is a G_K -stable finite free sub $B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} A$ -module of $W_{\text{dR}} := B_{\text{dR}} \otimes_{B_e} W_e$ which generates W_{dR} as B_{dR} -module.

We define the rank of W as the rank of W_e as $B_e \otimes_{\mathbb{Q}_p} A$ -module.

Later we just call A - B -pair if there is no risk of confusing about K .

Remark 2.2. The functor $V_A \mapsto W(V_A) := (B_e \otimes_{\mathbb{Q}_p} V_A, B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} V_A)$ from the category of A -representations of G_K to the category of A - B -pairs is exact and fully faithful because of the fact that $B_e \cap B_{\text{dR}}^+ = \mathbb{Q}_p$.

Proposition 2.3. *There exists a canonical bijection $\delta \mapsto W(\delta)$ between the set of continuous homomorphisms $\delta : K^\times \rightarrow A^\times$ and the set of isomorphism classes of rank one A - B -pairs.*

Proof. See proposition 2.15 of [Na10] □

Remark 2.4. This bijection is compatible with local class field theory, i.e. for any character $\tilde{\delta} : G_K^{\text{ab}} \rightarrow A^\times$ we have an isomorphism $W(\tilde{\delta} \circ \text{rec}_K) \xrightarrow{\sim} W(A(\tilde{\delta}))$.

Remark 2.5. For any $\lambda \in A^\times$, we define a continuous homomorphism $\delta_\lambda : K^\times \rightarrow A^\times$ such that $\delta_\lambda|_{\mathcal{O}_K^\times} = 1$ and $\delta_\lambda(\pi_K) = \lambda$. Then, $W(\delta_\lambda)$ is a crystalline A - B -pair corresponding to an A -filtered φ -module $D_\lambda := K_0 \otimes_{\mathbb{Q}_p} A e_\lambda$ such that $\varphi^f(e_\lambda) = \lambda e_\lambda$ and $\text{Fil}^0(K \otimes_{K_0} D_\lambda) = K \otimes_{K_0} D_\lambda$ and $\text{Fil}^1(K \otimes_{K_0} D_\lambda) = 0$.

Definition 2.6. Let W be an A - B -pair of rank d . We say that W is a split trianguline A - B -pair if there exists a filtration $\mathcal{T} : 0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{d-1} \subseteq W_d = W$ such that W_i are A - B -pairs for any i and that W_i/W_{i-1} are rank one A - B -pairs for any i . We call \mathcal{T} an A -triangulation of W . We say that the set $\{\delta_i\}_{i=1}^d$ ($\delta_i : K^\times \rightarrow A^\times$) is the parameter of \mathcal{T} if $W_i/W_{i-1} \xrightarrow{\sim} W(\delta_i)$ for any i .

Let V be an A -representation. We say that V is a split trianguline A -representation if $W(V)$ is a split trianguline A - B -pair.

In [BeCh09] and [Ch09b] ($K = \mathbb{Q}_p$ -case) and in [Na10] (any K -case), we study deformation theory of trianguline B -pairs (or trianguline (φ, Γ) -modules over the Robba ring), which we review below.

Let V be an E -representation of rank d and $A \in \mathcal{C}_E$. We say that the pair (V_A, ψ_A) is a deformation of V over A if V_A is an A -representation and $\psi_A : V_A \otimes_A A/\mathfrak{m}_A \xrightarrow{\sim} V$ is an isomorphism of E -representations. Let (V_A, ψ_A) and (V'_A, ψ'_A) be two deformations of V over A , we say that (V_A, ψ_A) and (V'_A, ψ'_A) are equivalent if there exists an isomorphism $f : V_A \xrightarrow{\sim} V'_A$ of A -representations such that $\psi_A = \psi'_A \circ (f \otimes_A \text{id}_{A/\mathfrak{m}_A})$. We define a functor $D_V : \mathcal{C}_E \rightarrow (\text{Sets})$ by

$$D_V(A) := \{ \text{equivalent classes of deformations of } V \text{ over } A \}$$

for any $A \in \mathcal{C}_E$.

Next, we consider the pair (V, \mathcal{T}) where V is a split trianguline E -representation with a triangulation $\mathcal{T} : 0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_d = W(V)$. For $A \in \mathcal{C}_E$, we say that the triple $(V_A, \psi_A, \mathcal{T}_A)$ is a trianguline deformation of (V, \mathcal{T}) over A if (V_A, ψ_A) is a deformation of V over A and $\mathcal{T}_A : 0 \subseteq W_{1,A} \subseteq W_{2,A} \subseteq \cdots \subseteq W_{d,A} = W(V_A)$ is an A -triangulation of V_A such that $W(\psi_A)(W_{i,A} \otimes_A A/\mathfrak{m}_A) = W_i$ for any $1 \leq i \leq d$, where $W(\psi_A) : W(V_A) \otimes_A A/\mathfrak{m}_A \xrightarrow{\sim} W(V)$ is the isomorphism induced from ψ_A . We say that two trianguline deformations $(V_A, \psi_A, \mathcal{T}_A)$ and $(V'_A, \psi'_A, \mathcal{T}'_A)$ over A are equivalent if there exists an isomorphism $f : V_A \xrightarrow{\sim} V'_A$ of A -representations such that $\psi_A = \psi'_A \circ (f \otimes_A \text{id}_{A/\mathfrak{m}_A})$ and $W(f)(W_{i,A}) = W'_{i,A}$ for any $1 \leq i \leq d$. We define a functor $D_{V,\mathcal{T}} \rightarrow (\text{Sets})$ by

$$D_{V,\mathcal{T}}(A) := \{ \text{equivalent classes of trianguline deformations of } (V, \mathcal{T}) \text{ over } A \}$$

for any $A \in \mathcal{C}_E$. We have a morphism of functors $D_{V,\mathcal{T}} \rightarrow D_V$ defined by $(V_A, \psi_A, \mathcal{T}_A) \mapsto (V_A, \psi_A)$. If D_V and $D_{V,\mathcal{T}}$ are represented by R_V and $R_{V,\mathcal{T}}$, then this morphism is given by a map $R_V \rightarrow R_{V,\mathcal{T}}$, which is surjection in many cases. For the representability and other properties of $D_{V,\mathcal{T}}$, we have the following proposition.

Proposition 2.7. *Let V be a split trianguline E -representation with a triangulation \mathcal{T} whose parameter is $\{\delta_i\}_{i=1}^d$. We assume that (V, \mathcal{T}) satisfies the following conditions,*

- (i) $\text{End}_{E[G_K]}(V) = E$ (then D_V is represented by R_V),
- (ii) For any $i < j$, $\delta_j/\delta_i \neq \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\leq 0}$,

then the functor $D_{V,\mathcal{T}}$ is represented by a quotient $R_{V,\mathcal{T}}$ of R_V . Moreover, if (V, \mathcal{T}) satisfies a following additional condition,

- (iii) For any $i < j$, $\delta_i/\delta_j \neq |N_{K/\mathbb{Q}_p}|_p \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\geq 1}$,

then $R_{V,\mathcal{T}}$ is formally smooth over E of dimension $[K : \mathbb{Q}_p] \frac{d(d+1)}{2} + 1$.

Proof. See [BeCh09] and corollary 2.30, lemma 2.48 and proposition 2.39 of [Na10]. \square

Next, we recall some relations between crystalline representations and trianguline representations. Let V be a crystalline E -representation of rank d . First,

we define the crystalline deformation functor D_V^{cris} which is a sub functor of D_V defined by

$$D_V^{\text{cris}}(A) := \{[(V_A, \psi_A)] \in D_V(A) | V_A \text{ is crystalline} \}$$

for any $A \in \mathcal{C}_E$. It is known that the natural inclusion $D_V^{\text{cris}} \hookrightarrow D_V$ is relatively representable and D_V^{cris} is formally smooth.

Let $D_{\text{cris}}(V) := (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ be the filtered φ -module associated to V , which is a finite free $K_0 \otimes_{\mathbb{Q}_p} E$ -module of rank d . We assume that the eigenvalues of φ^f ($f := [K_0 : \mathbb{Q}_p]$) $\{\alpha_1, \alpha_2, \dots, \alpha_d\} (\subseteq \overline{E})$ on $D_{\text{cris}}(V) \otimes_{K_0 \otimes_{\mathbb{Q}_p} E, \sigma \otimes \text{id}_E} \overline{E}$ ($\sigma : K_0 \hookrightarrow \overline{E}$) (this does not depend on the choice of σ) satisfies that $\alpha_i \neq \alpha_j$ for any $i \neq j$. Moreover, we assume that (or making E sufficiently large so that) $\{\alpha_1, \dots, \alpha_d\} \subseteq E$ and that $D_{\text{cris}}(V)$ can be written as $D_{\text{cris}}(V) = K_0 \otimes_{\mathbb{Q}_p} Ee_1 \oplus K_0 \otimes_{\mathbb{Q}_p} Ee_2 \oplus \dots \oplus K_0 \otimes_{\mathbb{Q}_p} Ee_d$ such that $K_0 \otimes_{\mathbb{Q}_p} Ee_i$ is φ -stable and that $\varphi^f(e_i) = \alpha_i e_i$ for any $1 \leq i \leq d$. Let \mathfrak{S}_d be the d -th permutation group. Under these assumptions, we define $(d!)$ -filtrations $\mathcal{F}_\tau : 0 \subseteq D_{\tau,1} \subseteq D_{\tau,2} \subseteq \dots \subseteq D_{\tau,d} = D_{\text{cris}}(V)$ on $D_{\text{cris}}(V)$ for any $\tau \in \mathfrak{S}_d$ by $D_{\tau,i} := \bigoplus_{j=1}^i K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(j)}$ (whose filtration is induced from that on $D_{\text{cris}}(V)$) for any $1 \leq i \leq d$. By the equivalence between the category of E -filtered φ -modules and the category of crystalline E - B -pairs (see [Be08] or [Na09], [Na10]), we obtain triangulations

$$\mathcal{T}_\tau : 0 \subseteq W_{\tau,1} \subseteq W_{\tau,2} \subseteq \dots \subseteq W_{\tau,d} = W(V)$$

such that $W_{\tau,i}$ are crystalline E - B -pairs and that $D_{\text{cris}}(W_{\tau,i}) \xrightarrow{\sim} D_{\tau,i}$ for any $1 \leq i \leq d$ for any $\tau \in \mathfrak{S}_d$. We recall the definition of benign representation in [Na10] (or generic crystalline representation in [Ch09b]) whose deformation theoretic property plays a crucial role in the proof of the main theorems of this article. Let $\{k_{1,\sigma}, k_{2,\sigma}, \dots, k_{d,\sigma}\}_{\sigma \in \mathcal{P}}$ be the Hodge-Tate weight of V such that $k_{1,\sigma} \geq k_{2,\sigma} \geq \dots \geq k_{d,\sigma}$ for any $\sigma \in \mathcal{P}$, in this article we define Hodge-Tate weight of the p -adic cyclotomic character $\chi_p : G_K \rightarrow E^\times$ by $\{1\}_{\sigma \in \mathcal{P}}$.

Definition 2.8. Let V be a crystalline representation satisfying all the above conditions. We say that V is benign if V satisfies the following conditions,

- (1) For any $i \neq j$, $\alpha_i \neq \alpha_j, p^{\pm f} \alpha_j$.
- (2) For any $\sigma \in \mathcal{P}$, $k_{1,\sigma} > k_{2,\sigma} > \dots > k_{d,\sigma}$.
- (3) For any $\tau \in \mathfrak{S}_d$ and $1 \leq i \leq d$, Hodge-Tate of $W_{\tau,i}$ is $\{k_{1,\sigma}, k_{2,\sigma}, \dots, k_{i,\sigma}\}_{\sigma \in \mathcal{P}}$.

If V is benign, then (V, \mathcal{T}_τ) satisfies all the properties in proposition 2.7 for any $\tau \in \mathfrak{S}_d$, hence the functor D_V is represented by R_V and the functors D_{V, \mathcal{T}_τ} are represented by R_{V, \mathcal{T}_τ} which are quotients of R_V . For $R_* = R_V, R_{V, \mathcal{T}_\tau}$, we define the tangent space of R_* by

$$t_{R_*} := \text{Hom}_E(\mathfrak{m}_{R_*} / \mathfrak{m}_{R_*}^2, E)$$

where \mathfrak{m}_{R_*} is the maximal ideal of R_* . Hence, we obtain a natural inclusion $t_{R_{V, \mathcal{T}_\tau}} \hookrightarrow t_{R_V}$ for any $\tau \in \mathfrak{S}_d$.

The following theorem is a crucial theorem for the proof of the main theorems of this article, which was first discovered by Chenevier (theorem 3.19 of [Ch09b]).

Theorem 2.9. *Let V be a benign representation of rank d , then we have an equality*

$$\sum_{\tau \in \mathfrak{S}_d} t_{R_V, \tau} = t_{R_V}.$$

Proof. See theorem 3.19 of [Ch09b] and theorem 2.61 of [Na10]. □

3. CONSTRUCTION OF FINITE SLOPE SUBSPACE

This section is the technical heart of this article. We generalize Kisin's construction of finite slope subspace X_{fs} (chapter 5 of [Ki03]) for any dimensional case and for any p -adic field case.

Let X be a separated rigid analytic space over E in the sense of Tate. For a point $x \in X$, we denote by $E(x)$ the residue field of X at x , which is a finite extension of E . We recall some terminologies which are used in [Ki03]. We say that an admissible open set $U \subseteq X$ is scheme theoretically dense in X if there exists an admissible affinoid covering $\{X_i := \mathrm{Spm}(R_i)\}_{i \in I}$ of X such that $U \cap X_i$ is associated to a dense Zariski open $U_i \subseteq \mathrm{Spec}(R_i)$ for any $i \in I$. For an invertible function $Y \in \mathcal{O}_X^\times$ and for an E -affinoid algebra R , we say that an E -morphism $f : \mathrm{Spm}(R) \rightarrow X$ is Y -small if there exists a finite extension E' of E and an element $\lambda \in (R \otimes_E E')^\times$ such that $E'[\lambda] \subseteq R \otimes_E E'$ is a finite étale E' -algebra and $\frac{Y}{\lambda}$ is topologically nilpotent in $R \otimes_E E'$. For any $f \in \Gamma(X, \mathcal{O}_X)$, we denote by $X_f := \{x \in X \mid f(x) \neq 0\}$ the Zariski open in X on which f does not have zero.

For any finite free \mathcal{O}_X -module M with a continuous \mathcal{O}_X -linear G_K -action, we denote by $M(x)$ the fiber of M at x , which is an $E(x)$ -representation of G_K and we denote by M^\vee the \mathcal{O}_X -dual of M . For such M , we can define Sen's polynomial

$$P_M(T) \in K \otimes_{\mathbb{Q}_p} \mathcal{O}_X[T]$$

which is a monic polynomial of degree n (where n is the \mathcal{O}_X -rank of M) such that, for any $x \in X$, $P_M(T)(x)$ the fiber at x is equal to $P_{M(x)}(T) \in K \otimes_{\mathbb{Q}_p} E(x)[T]$ Sen's polynomial of $M(x)$ (see [Ki03] (2.2)). Using the canonical decomposition

$$K \otimes_{\mathbb{Q}_p} \mathcal{O}_X[T] \xrightarrow{\sim} \prod_{\sigma \in \mathcal{P}} \mathcal{O}_X[T] : a \otimes f(T) \mapsto (\sigma(a)f(T))_{\sigma \in \mathcal{P}},$$

we decompose $P_M(T)$ into

$$P_M(T) = (P_{M, \sigma}(T))_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathcal{O}_X[T].$$

Let $d \in \mathbb{Z}_{\geq 1}$ be a positive integer. For any $1 \leq i \leq d$, let M_i be a finite free \mathcal{O}_X -module with a continuous \mathcal{O}_X -linear G_K -action. We assume that $P_{M_i, \sigma}(T)$ can be written as

$$P_{M_i, \sigma}(T) = TQ_{i, \sigma}(T)$$

for some monic polynomial $Q_{i,\sigma}(T) \in \mathcal{O}_X[T]$ for any $1 \leq i \leq d$ and $\sigma \in \mathcal{P}$. For any $1 \leq i \leq d$, let $Y_i \in \mathcal{O}_X^\times$ be an invertible function on X .

Under this situation, we prove the following theorem, which is the generalization of proposition 5.4 of [Ki03] or theorem 3.9 of [Na10] for any dimensional and any p -adic field case.

Theorem 3.1. *Under the above situation, there exists unique Zariski closed subspace $X_{fs} \subseteq X$ satisfying the following conditions (1) and (2).*

- (1) *For any $1 \leq i \leq d$ and $\sigma \in \mathcal{P}$ and $j \in \mathbb{Z}_{\leq 0}$, the subset $X_{fs, Q_{i,\sigma}(j)}$ is scheme theoretically dense in X_{fs} .*
- (2) *For any E -morphism $f : \text{Spm}(R) \rightarrow X$ which is Y_i -small for any $1 \leq i \leq d$ and factors through $X_{Q_{i,\sigma}(j)}$ for any $1 \leq i \leq d$ and $\sigma \in \mathcal{P}$ and $j \in \mathbb{Z}_{\leq 0}$, the following conditions (i) and (ii) are equivalent.*
 - (i) *f factors through $f : \text{Spm}(R) \rightarrow X_{fs} \hookrightarrow X$.*
 - (ii) *For any $1 \leq i \leq d$, any R -linear G_K -equivariant map*

$$h : f^*(M_i^\vee) \rightarrow B_{\text{dR}}^+ \hat{\otimes}_{\mathbb{Q}_p} R$$

factors through the canonical inclusion

$$K \otimes_{K_0} (B_{\text{max}}^+ \hat{\otimes}_{\mathbb{Q}_p} R)^{\varphi^f = Y_i} \hookrightarrow B_{\text{dR}}^+ \hat{\otimes}_{\mathbb{Q}_p} R.$$

Proof. For some notation which we use in this proof, see [Na10]. The proof of uniqueness is same as that of proposition 5.4 of [Ki03] or theorem 3.9 of [Na10].

By the same argument as in [Ki03] or [Na10], it suffices to construct X_{fs} when $X = \text{Spm}(R)$ is an affinoid which satisfies $|Y_i| |Y_i^{-1}| < \frac{1}{|\pi_K|_p}$ for any $1 \leq i \leq d$, where $|\cdot| : R \rightarrow \mathbb{Q}_{\geq 0}$ is an E -Banach norm on R . Then, we construct X_{fs} as follows. For any $1 \leq i \leq d$ and $\lambda_i \in \overline{E}$ such that $|Y_i^{-1}|^{-1} \leq |\lambda_i| \leq |Y_i|$, we take a finite Galois extension E' of E which contains λ_i and take a sufficiently large integer k such that the natural map

$$(B_{\text{max}, K}^+ \otimes_{K, \sigma} E')^{\varphi_K = \sigma(\pi_K)\lambda_i} \hookrightarrow B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \otimes_{K, \sigma} E'$$

is injection with a closed image for any $\sigma \in \mathcal{P}$ (this is possible by corollary 3.5 of [Na10]). Let $U_{i,\sigma}$ be the cokernel of this map, then $U_{i,\sigma}$ is also an E' -Banach space and we fix an orthonormalizable basis $\{e_{i,\sigma,j}\}_{j \in J_\sigma}$ of $U_{i,\sigma}$. Then, for any R -linear G_K -morphism

$$h : M_i^\vee \rightarrow B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} R$$

and $x \in \tilde{\mathbb{E}}^+$ such that $v(x) > 0$, we denote by

$$h_x : M_i^\vee \rightarrow U_{i,\sigma} \hat{\otimes}_{E'} (R \otimes_E E')$$

the composition of h with

$$B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} R \rightarrow B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} (R \otimes_E E') : y \mapsto P(x, \frac{Y_i}{\sigma(\pi_K)\lambda_i})y$$

and the natural quotient map

$$B_{\text{dR}}^+/t^k B_{\text{dR}}^+ \hat{\otimes}_{K,\sigma} (R \otimes_E E') \rightarrow U_{i,\sigma} \hat{\otimes}_{E'} (R \otimes_E E')$$

where $P(x, \frac{Y_i}{\sigma(\pi_K)\lambda_i})$ is defined by

$$P(x, \frac{Y_i}{\sigma(\pi_K)\lambda_i}) := \sum_{n \in \mathbb{Z}} \varphi_K([x])^n \otimes (\frac{Y_i}{\sigma(\pi_K)\lambda_i})^n \in (B_{\text{max},K}^+ \hat{\otimes}_{K,\sigma} (R \otimes_E E'))^{\varphi_K = \frac{\sigma(\pi_K)\lambda_i}{Y_i}}$$

whose convergence is proved in the proof of theorem 3.9 of [Na10]. Then, for any $m \in M_i^\vee$, we can write uniquely as $h_x(m) = \sum_{j \in J_\sigma} a(h, x, \lambda_i, m)_j e_{i,\sigma,j}$ for some $a(h, x, \lambda_i, m)_j \in R \otimes_E E'$. We define an ideal $I(h, x, \lambda_i)'$ of $R \otimes_E E'$ which is generated by $a(h, x, \lambda_i, m)_j$ for all $m \in M_i^\vee$ and $j \in J_\sigma$. Because $I(h, x, \tau(\lambda_i))' = \tau(I(h, x, \lambda_i)')$ for any $\tau \in \text{Gal}(E'/E)$, the ideal $\sum_{\tau \in \text{Gal}(E'/E)} I(h, x, \tau(\lambda_i))'$ descends to an ideal $I(h, x, \lambda_i) \subseteq R$. We define an ideal $I := \sum_{i,h,x,\lambda_i} I(h, x, \lambda_i) \subseteq R$. Finally, we define the smallest ideal I' so that I' contains I and the natural map $R/I' \rightarrow R/I'[\frac{1}{Q_{i,\sigma(j)}}]$ is injection for any $1 \leq i \leq d$ and $\sigma \in \mathcal{P}$ and $j \in \mathbb{Z}_{\leq 0}$. Then, the closed sub variety $\text{Spm}(R/I')$ satisfies the conditions (1) and (2), which we can prove in the same way as in the proof of proposition 5.4 of [Ki03] or theorem 3.9 of [Na10]. \square

Next, we prove a proposition concerning to some important properties of X_{fs} , which is a generalization of proposition 5.14 of [Ki03] and proposition 3.14 of [Na10].

Proposition 3.2. *In the above situation, let $U = \text{Spm}(R)$ be an affinoid open of X_{fs} which is Y_i -small for any $1 \leq i \leq d$. Let $k \in \mathbb{Z}_{\geq 1}$ be a sufficiently large integer such that, for any $\sigma \in \mathcal{P}$ and $1 \leq i \leq d$, there exists a short exact sequence,*

$$0 \rightarrow (B_{\text{max},K}^+ \hat{\otimes}_{K,\sigma} R)^{\varphi_K = Y_i} \rightarrow B_{\text{dR}}^+/t^k B_{\text{dR}}^+ \hat{\otimes}_{K,\sigma} R \rightarrow U_{i,\sigma} \rightarrow 0$$

for a Banach R -module with property (Pr) $U_{i,\sigma}$ (this is possible by proposition 3.7 of [Na10]), then the following hold.

- (1) For any $1 \leq i \leq d$ and $\sigma \in \mathcal{P}$, the natural injection

$$(B_{\text{max},K}^+ \hat{\otimes}_{K,\sigma} (M_i \otimes_{\mathcal{O}_X} R))^{G_K, \varphi_K = Y_i} \hookrightarrow (B_{\text{dR}}^+/t^k B_{\text{dR}}^+ \hat{\otimes}_{K,\sigma} (M_i \otimes_{\mathcal{O}_X} R))^{G_K}$$

is isomorphism.

- (2) For any $1 \leq i \leq d$ and $\sigma \in \mathcal{P}$, let $H_{i,\sigma} \subseteq R$ be the smallest ideal such that any G_K -equivariant R -linear map

$$h : M_i^\vee \rightarrow B_{\text{dR}}^+/t^k B_{\text{dR}}^+ \hat{\otimes}_{K,\sigma} R$$

factors through

$$B_{\text{dR}}^+/t^k B_{\text{dR}}^+ \hat{\otimes}_{K,\sigma} H_{i,\sigma} \hookrightarrow B_{\text{dR}}^+/t^k B_{\text{dR}}^+ \hat{\otimes}_{K,\sigma} R,$$

and put $H := \prod_{1 \leq d, \sigma \in \mathcal{P}} H_{i,\sigma} \subseteq R$, then $\text{Spm}(R) \setminus V(H)$ and $\text{Spm}(R) \setminus V(H_{i,\sigma})$ are scheme theoretically dense in $\text{Spm}(R)$.

- (3) For any point $x \in \text{Spm}(R)$ and for any $1 \leq i \leq d$ and $\sigma \in \mathcal{P}$, $(B_{\max, K}^+ \otimes_{K, \sigma} M_i(x))^{G_K, \varphi_K = Y_i}$ is not zero.

Proof. The proof is essentially same as that of proposition 3.14 of [Na10]. First we prove (1). By the definition of X_{fs} , we have an equality

$$(B_{\max, K}^+ \otimes_{K, \sigma} (M_i \otimes_{\mathcal{O}_X} R_x / \mathfrak{m}_x^n))^{G_K, \varphi_K = Y_i} = (B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \otimes_{K, \sigma} (M_i \otimes_{\mathcal{O}_X} R_x / \mathfrak{m}_x^n))^{G_K}$$

for any $x \in \text{Spm}(R)$ such that $Q_{i, \sigma}(j)(x) \neq 0$ for any $\sigma \in \mathcal{P}$ and $1 \leq i \leq d$ and $j \in \mathbb{Z}_{\leq 0}$, where we denote by R_x the localization of R at \mathfrak{m}_x . Hence it suffices to show that the natural map $R \rightarrow \prod_{x \in V, n \geq 1} R_x / \mathfrak{m}_x^n$ is injection where we define

$$V := \{x \in \text{Spm}(R) | Q_{i, \sigma}(j)(x) \neq 0 \text{ for any } \sigma \in \mathcal{P}, 1 \leq i \leq d, j \in \mathbb{Z}_{\leq 0}\}.$$

Let $f \in R$ be an element in the kernel of this map. If we denote by W the support of f in $\text{Spm}(R)$, then we have $W \subseteq \cup_{\sigma \in \mathcal{P}, 1 \leq i \leq d, j \in \mathbb{Z}_{\leq 0}} V(Q_{i, \sigma}(j))$, where, for any $Q \in R$, we denote by $V(Q)$ the reduced closed subspace of $\text{Spm}(R)$ such that $V(Q) = \{x \in \text{Spm}(R) | Q(x) = 0\}$. Then, by lemma 5.7 of [Ki03], there exists a Q which is a finite product of $Q_{i, \sigma}(j)$ such that $W \subseteq V(Q)$, so we have $\text{Spm}(R)_Q \subseteq X \setminus W$, in particular f is zero in $R[\frac{1}{Q}]$. Because $\text{Spm}(R)_Q$ is scheme theoretically dense in $\text{Spm}(R)$, then we have $f = 0$, this proves (1).

Next we prove (2). First, we show that if $x \in V$ then $x \in \text{Spm}(R) \setminus V(H_{i, \sigma})$ for any $1 \leq i \leq d$ and $\sigma \in \mathcal{P}$ (then we also have $x \in \text{Spm}(R) \setminus V(H)$). If $x \in V \cap V(H_{i, \sigma})$ for some $1 \leq i \leq d$ and $\sigma \in \mathcal{P}$, then we have an equality $(B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \otimes_{K, \sigma} (M_i \otimes_{\mathcal{O}_X} R))^{G_K} \otimes_R R_x / \mathfrak{m}_x = (B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \otimes_{K, \sigma} (M_i \otimes_{\mathcal{O}_X} R_x / \mathfrak{m}_x))^{G_K}$ which is a one dimensional $E(x)$ -vector space by corollary 2.6 of [Ki03], on the other hands this is zero because $H_{i, \sigma} \subseteq \mathfrak{m}_x$ and by the definition of $H_{i, \sigma}$, this is a contradiction. Hence, by the same argument as in the proof of (1), there exists Q which is a finite product of $Q_{i, \sigma}(j)$ such that $\text{Spm}(R)_Q \subseteq \text{Spm}(R) \setminus V(H)$. By the definition of X_{fs} , this shows that $\text{Spm}(R) \setminus V(H)$ is scheme theoretically dense in $\text{Spm}(R)$ and then $\text{Spm}(R) \setminus V(H_{i, \sigma})$ are also scheme theoretically dense.

Using (2), we can prove (3) in the same way as that of proposition 3.14 of [Na10]. \square

We apply theorem 3.1 to the following set up. Let \overline{V} be an \mathbb{F} -representation of G_K of dimension d . Let $\mathcal{C}_{\mathcal{O}}$ be the category of Artin local \mathcal{O} -algebra A such that $A/\mathfrak{m}_A \xrightarrow{\sim} \mathbb{F}$. For any $A \in \mathcal{C}_{\mathcal{O}}$, we say that a couple (V_A, ψ_A) is a deformation of \overline{V} over A if V_A is an A -representation of G_K and $\psi_A : V_A \otimes_A A/\mathfrak{m}_A \xrightarrow{\sim} \overline{V}$ is an isomorphism of \mathbb{F} -representations. We say that two deformations (V_A, ψ_A) and (V'_A, ψ'_A) of \overline{V} over A are equivalent if there exists an isomorphism $f : V_A \xrightarrow{\sim} V'_A$ of A -representations such that $\psi_A = \psi'_A \circ (f \otimes \text{id}_{A/\mathfrak{m}_A})$. We consider a functor

$$D_{\overline{V}} : \mathcal{C}_{\mathcal{O}} \rightarrow (\text{Sets})$$

defined by

$$D_{\overline{V}}(A) := \{ \text{equivalent classes of deformations of } \overline{V} \text{ over } A \}$$

for any $A \in \mathcal{C}_{\mathcal{O}}$. In this paper, for simplicity we assume that \overline{V} satisfies that

$$\mathrm{End}_{\mathbb{F}[G_K]}(\overline{V}) = \mathbb{F}.$$

Under this assumption, the functor $D_{\overline{V}}$ is pro-represented by the universal deformation ring $R_{\overline{V}}$. Let V^{univ} be the universal deformation of \overline{V} over $R_{\overline{V}}$. Let $\mathfrak{X}_{\overline{V}}$ be the rigid analytic space over E associated to the formal \mathcal{O} -scheme $\mathrm{Spf}(R_{\overline{V}})$. Then, V^{univ} naturally defines a finite free $\mathcal{O}_{\mathfrak{X}_{\overline{V}}}$ -module \mathcal{M} of rank d with a continuous $\mathcal{O}_{\mathfrak{X}_{\overline{V}}}$ -linear G_K -action. Let $P_{\mathcal{M}}(T) := (P_{\mathcal{M},\sigma}(T))_{\sigma \in \mathcal{P}} \in K \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X}_{\overline{V}}}[T] \xrightarrow{\sim} \bigoplus_{\sigma \in \mathcal{P}} \mathcal{O}_{\mathfrak{X}_{\overline{V}}}[T]$ be Sen's polynomial of \mathcal{M} .

Let $\det(\mathcal{M}) : G_K^{\mathrm{ab}} \rightarrow \mathcal{O}_{\mathfrak{X}_{\overline{V}}}^{\times}$ be the determinant character of \mathcal{M} . We also write by the same letter the continuous homomorphism $\det(\mathcal{M}) : K^{\times} \rightarrow \mathcal{O}_{\mathfrak{X}_{\overline{V}}}^{\times}$ defined by $\det(\mathcal{M}) \circ \mathrm{rec}_K$.

Next we recall the definition of weight space for K . Let \mathcal{W} and \mathcal{T} be the functors from the category of rigid analytic spaces over E to the category of abelian groups defined by

$$\mathcal{W}(X') := \{\delta : \mathcal{O}_K^{\times} \rightarrow \Gamma(X', \mathcal{O}_{X'})^{\times} \mid \delta \text{ is a continuous homomorphism}\}$$

and

$$\mathcal{T}(X') := \{\delta : K^{\times} \rightarrow \Gamma(X', \mathcal{O}_{X'})^{\times} \mid \delta \text{ is a continuous homomorphism}\}$$

for any rigid analytic space X' over E . It is known that \mathcal{W} and \mathcal{T} are representable and \mathcal{W} is represented by the rigid analytic group variety associated to the Iwasawa algebra $\mathcal{O}[[\mathcal{O}_K^{\times}]]$ and \mathcal{W} is (as a rigid space) $\sharp(\mathcal{O}_{K,\mathrm{tor}}^{\times})$ -union of d -dimensional open unit discs and, if we fix a uniformizer $\pi_K \in \mathcal{O}_K$, we have an isomorphism

$$\mathcal{T} \xrightarrow{\sim} \mathcal{W} \times_E \mathbb{G}_{m/E}^{\mathrm{an}} : \delta \mapsto (\delta|_{\mathcal{O}_K^{\times}}, \delta(\pi_K)).$$

We denote the projections by

$$p_1 : \mathcal{T} \rightarrow \mathcal{W} : \delta \mapsto \delta|_{\mathcal{O}_K^{\times}}, p_{2,\pi_K} : \mathcal{T} \rightarrow \mathbb{G}_{m/E}^{\mathrm{an}} : \delta \mapsto \delta(\pi_K).$$

Let $Y \in \Gamma(\mathbb{G}_{m/E}^{\mathrm{an}}, \mathcal{O}_{\mathbb{G}_{m/E}^{\mathrm{an}}})^{\times}$ be the canonical coordinate. Let

$$\delta_{\mathcal{W}}^{\mathrm{univ}} : \mathcal{O}_K^{\times} \rightarrow \Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}})^{\times}$$

be the universal homomorphism of the functor \mathcal{W} , which is equal to the composition of the canonical map $\mathcal{O}_K^{\times} \rightarrow \mathcal{O}[[\mathcal{O}_K^{\times}]]^{\times} : a \mapsto [a]$ with the canonical map $\mathcal{O}[[\mathcal{O}_K^{\times}]]^{\times} \rightarrow \Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}})^{\times}$. Let $\tilde{\delta}_{\mathcal{W}}^{\mathrm{univ}} : G_K^{\mathrm{ab}} \rightarrow \Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}})^{\times}$ be the continuous character defined by $\tilde{\delta}_{\mathcal{W}}^{\mathrm{univ}} \circ \mathrm{rec}_K|_{\mathcal{O}_K^{\times}} = \delta_{\mathcal{W}}^{\mathrm{univ}}$ and $\tilde{\delta}_{\mathcal{W}}^{\mathrm{univ}}(\mathrm{rec}_K(\pi_K)) = 1$. Then, the universal homomorphism

$$\delta_{\mathcal{T}}^{\mathrm{univ}} : K^{\times} \rightarrow \Gamma(\mathcal{T}, \mathcal{O}_{\mathcal{T}})^{\times}$$

of the functor \mathcal{T} satisfies

$$\delta_{\mathcal{T}}^{\mathrm{univ}}|_{\mathcal{O}_K^{\times}} = p_1^* \circ \delta_{\mathcal{W}}^{\mathrm{univ}}, \delta_{\mathcal{T}}^{\mathrm{univ}}(\pi_K) = p_{2,\pi_K}^*(Y).$$

Because, for any E -affinoid algebra A , any continuous homomorphism $\delta : \mathcal{O}_K^\times \rightarrow A^\times$ is locally \mathbb{Q}_p -analytic (see proposition 8.3 of [Bu07]), hence for any rigid analytic space X' over E and for any continuous homomorphism $\delta : \mathcal{O}_K^\times \rightarrow \Gamma(X', \mathcal{O}_{X'})^\times$, we can define

$$\frac{\partial \delta(x)}{\partial \sigma(x)}|_{x=1} \in \Gamma(X', \mathcal{O}_{X'})$$

the partial differential of δ by $\sigma(x)$ ($\sigma \in \mathcal{P}$) at $x = 1 \in \mathcal{O}_K^\times$.

Here, we prove a proposition concerning to Hodge-Tate weight of $E(\delta)$ for any continuous characters $\delta : G_K^{\text{ab}} \rightarrow \mathcal{O}_E^\times$. We recall that $\chi_{\text{LT}} : G_K^{\text{ab}} \rightarrow \mathcal{O}_K^\times \hookrightarrow \mathcal{O}_E^\times$ is the Lubin-Tate character associated to a fixed uniformizer $\pi_K \in \mathcal{O}_K$.

Proposition 3.3. *Let A be an affinoid over E and let $\delta : G_K^{\text{ab}} \rightarrow A^\times$ be a continuous character. Then, the σ -part of Hodge-Tate weight of δ is equal to $\frac{\partial \delta(\text{rec}_K(x))}{\partial \sigma(x)}|_{x=1} \in A$.*

Proof. Let $\delta : G_K^{\text{ab}} \rightarrow A^\times$ be a continuous character. Because twisting by a unramified character does not change Hodge-Tate weight, we may assume that $\delta(\text{rec}_K(\pi)) = 1$. By the universality of \mathcal{W} , then there exists a morphism $f : \text{Spm}(A) \rightarrow \mathcal{W}$ such that $\delta = f^* \circ \tilde{\delta}_{\mathcal{W}}^{\text{univ}}$. Because $\frac{\partial \delta(\text{rec}_K(x))}{\partial \sigma(x)}|_{x=1}$ and Hodge-Tate weight of δ are compatible with base change, so it suffices to show that the σ -part of Hodge-Tate weight of $\tilde{\delta}_{\mathcal{W}}^{\text{univ}}$ is equal to $\frac{\partial \tilde{\delta}_{\mathcal{W}}^{\text{univ}}(x)}{\partial \sigma(x)}|_{x=1} \in \Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}})$. We denote by $a_\sigma \in \Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}})$ the σ -part of Hodge-Tate weight of $\tilde{\delta}_{\mathcal{W}}^{\text{univ}}$. If we put $\mathcal{W}_0 := \{\prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma} | \{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}\} \subseteq \mathcal{W}$, then a_σ is equal to $\frac{\partial \tilde{\delta}_{\mathcal{W}}^{\text{univ}}(x)}{\partial \sigma(x)}|_{x=1}$ at any points of \mathcal{W}_0 because, for any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}$, the character $\prod_{\sigma \in \mathcal{P}} \sigma(\chi_{\text{LT}})^{k_\sigma}$ is a crystalline character with Hodge-Tate weights $\{k_\sigma\}_{\sigma \in \mathcal{P}}$ by the result of Fontaine. Because \mathcal{W}_0 is Zariski dense in \mathcal{W} by lemma 2.7 of [Ch09a], hence a_σ is equal to $\frac{\partial \tilde{\delta}_{\mathcal{W}}^{\text{univ}}(x)}{\partial \sigma(x)}|_{x=1}$ on \mathcal{W} , which proves the proposition. \square

Let $Z := \mathfrak{X}_{\overline{V}} \times_E \mathcal{T}^{\times(d-1)}$. Then, any point $z \in Z$ can be written as $z = (x, \delta_1, \dots, \delta_{d-1})$ where $x \in \mathfrak{X}_{\overline{V}}$ and $\delta_i : K^\times \rightarrow E(z)^\times$. Let

$$p : Z \rightarrow \mathfrak{X}_{\overline{V}} : (x, \delta_1, \dots, \delta_{d-1}) \mapsto x$$

and, for $1 \leq i \leq d-1$,

$$q_i : Z \rightarrow \mathcal{T} : (x, \delta_1, \dots, \delta_{d-1}) \mapsto \delta_i$$

be the projections. We denote by $N := p^*(\mathcal{M})$, $P_{N,\sigma}(T) := p^*(P_{\mathcal{M},\sigma}(T)) \in \mathcal{O}_Z[T]$,

$$\delta_i^{\text{univ}} := q_i^* \circ \delta_{\mathcal{T}}^{\text{univ}} : K^\times \rightarrow \Gamma(\mathcal{T}, \mathcal{O}_{\mathcal{T}})^\times \rightarrow \Gamma(Z, \mathcal{O}_Z)^\times : a \mapsto q_i^*(\delta_{\mathcal{T}}^{\text{univ}}(a)),$$

$$Y_i := q_i^*(\delta_{\mathcal{T}}^{\text{univ}}(\pi_K)) \in \Gamma(Z, \mathcal{O}_Z)^\times.$$

For $i = d$, we define

$$\delta_d^{\text{univ}} := \det(N) / \prod_{i=1}^{d-1} \delta_i^{\text{univ}} : K^\times \rightarrow \Gamma(Z, \mathcal{O}_Z)^\times.$$

For any $1 \leq i \leq d-1$, we define a continuous character

$$\tilde{\delta}_i^{\text{univ}} : G_K^{\text{ab}} \rightarrow \Gamma(Z, \mathcal{O}_Z)^\times$$

such that

$$\tilde{\delta}_i^{\text{univ}} \circ \text{rec}_K|_{\mathcal{O}_K^\times} = \delta_i^{\text{univ}}|_{\mathcal{O}_K^\times} \text{ and } \tilde{\delta}_i^{\text{univ}}(\text{rec}_K(\pi_K)) = 1$$

and define

$$\delta_{Y_i} : K^\times \rightarrow \Gamma(Z, \mathcal{O}_Z)^\times$$

such that $\delta_{Y_i}|_{\mathcal{O}_K^\times} = 1$ and $\delta_{Y_i}(\pi_K) = Y_i$.

Under these notations, we define a Zariski closed subspace

$$Z_0 \subseteq Z$$

defined as the largest Zariski closed sub space such that the equalities

$$P_{N,\sigma}(T) = \prod_{i=1}^d (T - \frac{\partial \delta_i^{\text{univ}}(x)}{\partial \sigma(x)}|_{x=1})$$

hold on Z_0 for any $\sigma \in \mathcal{P}$, i.e. if we denote by

$$P_{N,\sigma}(T) - \prod_{i=1}^d (T - \frac{\partial \delta_i^{\text{univ}}(x)}{\partial \sigma(x)}|_{x=1}) := a_{d-1,\sigma} T^{d-1} + \dots + a_{0,\sigma},$$

then Z_0 is defined by the ideal generated by $\{a_{i,\sigma}\}_{0 \leq i \leq d-1, \sigma \in \mathcal{P}}$. For any $1 \leq i \leq d-1$, let $\wedge^i N$ be the i -th wedge product of N over \mathcal{O}_Z . On Z_0 , the σ -part of Hodge-Tate polynomial of

$$N_i := (\wedge^i N \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(\prod_{j=1}^i (\tilde{\delta}_j^{\text{univ}})^{-1}))|_{Z_0}$$

is written as $TQ_{i,\sigma}(T)$ for a monic polynomial $Q_{i,\sigma}(T) \in \mathcal{O}_{Z_0}[T]$ because the σ -part of Hodge-Tate weight of $\tilde{\delta}_i^{\text{univ}}$ is $\frac{\partial \delta_i^{\text{univ}}(x)}{\partial \sigma(x)}|_{x=1}$ by proposition 3.3. Hence, we can apply theorem 3.1 to this situation, precisely we obtain the following corollary.

Corollary 3.4. *Under the above situation, there exists a unique Zariski closed subspace*

$$\mathcal{E}_{\overline{V}} := Z_{0,fs} \subseteq Z_0$$

satisfying the following conditions (1) and (2).

- (1) For any $1 \leq i \leq d-1$ and $\sigma \in \mathcal{P}$ and $j \in \mathbb{Z}_{\leq 0}$, $\mathcal{E}_{\overline{V}, Q_{i,\sigma}(j)}$ is scheme theoretically dense in $\mathcal{E}_{\overline{V}}$.

- (2) For any E -morphism $f : \mathrm{Spm}(R) \rightarrow Z_0$ which is Y_i -small for all $1 \leq i \leq d-1$ and factors through $Z_{0, Q_{i, \sigma(j)}}$ for any $1 \leq i \leq d-1$ and $\sigma \in \mathcal{P}$ and $j \in \mathbb{Z}_{\leq 0}$, the following conditions (i) and (ii) are equivalent.

- (i) f factors through $\mathcal{E}_{\overline{V}}$.
- (ii) For any $1 \leq i \leq d-1$, any R -linear G_K -equivariant map

$$h : f^*(N_i^\vee) \rightarrow B_{\mathrm{dR}}^+ \hat{\otimes}_{\mathbb{Q}_p} R$$

factors through the natural inclusion

$$K \otimes_{K_0} (B_{\mathrm{max}}^+ \hat{\otimes}_{\mathbb{Q}_p} R)^{\varphi^f = \prod_{j=1}^i Y_j} \hookrightarrow B_{\mathrm{dR}}^+ \hat{\otimes}_{\mathbb{Q}_p} R.$$

Remark 3.5. By the definition, we can check that $f : \mathrm{Spm}(R) \rightarrow Z_0$ is Y_i -small for any $1 \leq i \leq d-1$ if and only if f is $(\prod_{j=1}^i Y_j)$ -small for any $1 \leq i \leq d-1$.

Let $x \in \mathfrak{X}_{\overline{V}}$ be a point of $\mathfrak{X}_{\overline{V}}$, then x corresponds to an $E(x)$ -representation V_x of G_K such that there exists a G_K -stable $\mathcal{O}_{E(x)}$ -lattice $T_x \subseteq V_x$ which satisfies $T_x / \pi_{E(x)} T_x \xrightarrow{\sim} \overline{V} \otimes_{\mathbb{F}} (\mathcal{O}_{E(x)} / \pi_{E(x)} \mathcal{O}_{E(x)})$. We assume that, for a finite extension E' of $E(x)$, $V_x \otimes_{E(x)} E'$ is a split trianguline E' -representation with a triangulation

$$\mathcal{T}_x : 0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_d := W(V_x \otimes_{E(x)} E').$$

We denote by $\{\delta_i\}_{i=1}^d$ the parameter of \mathcal{T}_x , i.e. $\delta_i : K^\times \rightarrow E'^\times$ satisfies $W_i / W_{i-1} \xrightarrow{\sim} W(\delta_i)$. By proposition 3.3, then the couple (V_x, \mathcal{T}_x) determines an E' -rational point

$$z := z_{(V_x, \mathcal{T}_x)} := (x, \delta_1, \delta_2, \dots, \delta_{d-1}) \in Z_0(E').$$

Moreover, by Galois descend, all these are in fact defined over $E(z) (\subseteq E')$, i.e. the E' -triangulation \mathcal{T}_x descend to an $E(z) (\subseteq E')$ -triangulation of $V_x \otimes_{E(x)} E(z)$ with the same parameter. Hence, in this article, if we write $z := z_{(V_x, \mathcal{T}_x)} \in Z_0$, we always assume that V_x is a split trianguline $E(z)$ -representation with a $E(z)$ -triangulation \mathcal{T}_x .

Proposition 3.6. Let (V_x, \mathcal{T}_x) be a couple as above which satisfies the following conditions (put $z := z_{(V_x, \mathcal{T}_x)}$),

- (0) $\mathrm{End}_{E(z)[G_K]}(V_x) = E(z)$
- (1) For any $1 \leq i < j \leq d$, $\delta_j / \delta_i \neq \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\leq 0}$.
- (2) For any $1 \leq i < j \leq d$, $\delta_i / \delta_j \neq |N_{K/\mathbb{Q}_p}|_p \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\geq 1}$.

Then, the point $z \in Z_0$ is contained in $\mathcal{E}_{\overline{V}}$.

Proof. First, by Lemma 3.10 of [Na10] (we can prove this lemma in d -dimensional case in the same way), we may assume that $E(z) = E$. We prove this proposition under this assumption. By the conditions (0) and (1) and by proposition 2.34 of [Na10], D_{V_x} and the trianguline deformation functor D_{V_x, \mathcal{T}_x} are represented by R_{V_x}

and by R_{V_x, τ_x} respectively. We denote by V_x^{univ} the universal deformation of V_x over R_{V_x} and

$$\mathcal{T}_x^{\text{univ}} : 0 \subseteq W_1^{\text{univ}} \subseteq W_2^{\text{univ}} \subseteq \cdots \subseteq W_d^{\text{univ}} = W(V_x^{\text{univ}}) \otimes_{R_{V_x}} R_{V_x, \tau_x}$$

the universal triangulation and denote by $\{\delta_{i,x}\}_{i=1}^d$ the parameter of $\mathcal{T}_x^{\text{univ}}$, i.e.

$$\delta_{i,x} : K^\times \rightarrow R_{V_x, \tau_x}^\times \text{ such that } W_i^{\text{univ}}/W_{i-1}^{\text{univ}} \xrightarrow{\sim} W(\delta_{i,x}).$$

Because we have canonical isomorphisms $\hat{\mathcal{O}}_{\mathfrak{X}_{\overline{V}}, x} \xrightarrow{\sim} R_{V_x}$ and $\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}_{\overline{V}}}} \hat{\mathcal{O}}_{\mathfrak{X}_{\overline{V}}, x} \xrightarrow{\sim} V_x^{\text{univ}}$ by proposition 9.5 of [Ki03], we can define a morphism

$$g : \hat{\mathcal{O}}_{Z,z} \xrightarrow{\sim} \hat{\mathcal{O}}_{\mathfrak{X}_{\overline{V}}, x} \hat{\otimes}_E (\hat{\otimes}_{i=1}^{d-1} \hat{\mathcal{O}}_{\mathcal{T}, \delta_i}) \xrightarrow{\sim} R_{V_x} \hat{\otimes}_E (\hat{\otimes}_{i=1}^{d-1} \hat{\mathcal{O}}_{\mathcal{T}, \delta_i}) \rightarrow R_{V_x, \tau_x} \hat{\otimes}_E (\hat{\otimes}_{i=1}^{d-1} \hat{\mathcal{O}}_{\mathcal{T}, \delta_i}).$$

We define an ideal I of $R_{V_x, \tau_x} \hat{\otimes}_E (\hat{\otimes}_{i=1}^{d-1} \hat{\mathcal{O}}_{\mathcal{T}, \delta_i})$ generated by

$$\{\delta_{i,x}(a) \otimes 1 - g(\delta_i^{\text{univ}}(a)) \mid 1 \leq i \leq d-1, a \in K^\times\}.$$

We denote by

$$R_z := (R_{V_x, \tau_x} \hat{\otimes}_E (\hat{\otimes}_{i=1}^{d-1} \hat{\mathcal{O}}_{\mathcal{T}, \delta_i})) / I.$$

Then, the natural map

$$R_{V_x, \tau_x} \rightarrow R_{V_x, \tau_x} \hat{\otimes}_E (\hat{\otimes}_{i=1}^d \hat{\mathcal{O}}_{\mathcal{T}, \delta_i}) \rightarrow R_z : y \mapsto \overline{y \otimes 1}$$

is an isomorphism and on R_z the universal parameter $\{\delta_{i,x}\}_{i=1}^d$ is equal to $\{\delta_i^{\text{univ}}\}_{i=1}^d$.

We define a morphism $h : \hat{\mathcal{O}}_{Z,z} \rightarrow R_z$ by

$$h : \hat{\mathcal{O}}_{Z,z} \xrightarrow{g} R_{V_x, \tau_x} \hat{\otimes}_E (\hat{\otimes}_{i=1}^{d-1} \hat{\mathcal{O}}_{\mathcal{T}, \delta_i}) \rightarrow R_z.$$

Because we have an isomorphism $\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}_{\overline{V}}}} \hat{\mathcal{O}}_{\mathfrak{X}_{\overline{V}}, x} \xrightarrow{\sim} V_x^{\text{univ}}$, $N \otimes_{\mathcal{O}_Z} R_z$ is isomorphic to the universal trianguline deformation of V_x over $R_{V_x, \tau_x} \xrightarrow{\sim} R_z$. Hence, the σ -part of Hodge-Tate polynomial of $N \otimes_{\mathcal{O}_Z} R_z$ is equal to $\prod_{i=1}^d (T - \frac{\partial \delta_i^{\text{univ}}(x)}{\partial \sigma(x)}|_{x=1})$, so the natural morphism

$$\text{Spm}(R_z/\mathfrak{m}^n) \rightarrow Z$$

factors through

$$f_n : \text{Spm}(R_z/\mathfrak{m}^n) \rightarrow Z_0$$

for any $n \geq 1$ where $\mathfrak{m} \subseteq R_z$ is the maximal ideal. We show that f_n also factors through $\mathcal{E}_{\overline{V}}$ for any n , which proves this proposition because the point of Z_0 determined by f_1 is equal to z .

We note that R_z is formally smooth over E , in particular is a domain by the condition (2) and by proposition 2.36 by [Na10]. We take an affinoid neighborhood $\text{Spm}(R)$ of $z \in Z_0$ as in the proof of theorem 3.1. Then, by the proof of theorem 3.1, it suffices to show the following lemma.

□

Lemma 3.7. *Under the above situation, the following hold.*

- (1) *For any $1 \leq i \leq d-1$ and $\sigma \in \mathcal{P}$ and $j \in \mathbb{Z}_{\leq 0}$, $Q_{i,\sigma}(j)$ is nonzero in R_z .*

(2) For any $1 \leq i \leq d-1$ and $\sigma \in \mathcal{P}$ and $k \in \mathbb{Z}_{\geq 1}$, the natural map

$$\varprojlim_n (B_{\max, K}^+ \otimes_{K, \sigma} f_n^*(N_i))^{G_K, \varphi_K = \prod_{j=1}^i Y_j} \rightarrow \varprojlim_n (B_{\mathrm{dR}}^+ / t^k B_{\mathrm{dR}}^+ \otimes_{K, \sigma} f_n^*(N_i))^{G_K}$$

is surjection.

Proof. If we can prove (i), then (ii) can be proved in the same way as in the proof of proposition 2.8 of [Ki03]. We prove (i). Because the σ -part of Hodge-Tate weight of $f_n^*(N)$ is

$$\left\{ \frac{\partial \delta_{1,x}(x)}{\partial \sigma(x)}|_{x=1}, \dots, \frac{\partial \delta_{d,x}(x)}{\partial \sigma(x)}|_{x=1} \right\},$$

the σ -part of Hodge-Tate weight of $f_n^*(N_i)$ is

$$\left\{ \sum_{k=1}^i \frac{\partial \delta_{j_k, x}(x)}{\partial \sigma(x)}|_{x=1} - \sum_{j=1}^i \frac{\partial \delta_{j, x}(x)}{\partial \sigma(x)}|_{x=1} \mid 1 \leq j_1 < j_2 < \dots < j_i \leq d \right\}$$

for $1 \leq i \leq d-1$. Hence, the σ -part of Hodge-Tate polynomial of $f_n^*(N_i)$ is equal to $\prod_{j_1 < \dots < j_i} (T - \sum_{k=1}^i \frac{\partial \delta_{j_k, x}(x)}{\partial \sigma(x)}|_{x=1} + \sum_{j=1}^i \frac{\partial \delta_{j, x}(x)}{\partial \sigma(x)}|_{x=1})$, so

$$Q_{i, \sigma}(T) = \prod_{(j_1, \dots, j_i) \neq (1, 2, \dots, i), j_1 < \dots < j_i} (T - \sum_{k=1}^i \frac{\partial \delta_{j_k, x}(x)}{\partial \sigma(x)}|_{x=1} + \sum_{j=1}^i \frac{\partial \delta_{j, x}(x)}{\partial \sigma(x)}|_{x=1}).$$

Therefore, it suffices to show the following lemma. □

Lemma 3.8. For any $1 \leq i \leq d-1$ and $j_1 < j_2 < \dots < j_i$ such that $(j_1, \dots, j_i) \neq (1, 2, \dots, i)$, then

$$\sum_{k=1}^i \frac{\partial \delta_{j_k, x}(x)}{\partial \sigma(x)}|_{x=1} - \sum_{j=1}^i \frac{\partial \delta_{j, x}(x)}{\partial \sigma(x)}$$

is not constant, i.e. not contained in E .

Proof. For any continuous homomorphism $\delta : \mathcal{O}_K^\times \rightarrow E^\times$, we define a functor $D_\delta : \mathcal{C}_E \rightarrow (\text{Sets})$ by

$$D_\delta(A) := \{\delta_A : \mathcal{O}_K^\times \rightarrow A^\times \mid \text{continuous homomorphisms such that } \delta_A \pmod{\mathfrak{m}_A} = \delta\},$$

for any $A \in \mathcal{C}_E$. Then, D_δ is represented by R_δ which is formally smooth over E of dimension $[K : \mathbb{Q}_p]$. If we denote by $\delta^{\mathrm{univ}} : \mathcal{O}_K^\times \rightarrow R_\delta^\times$ the universal deformation of δ , then the σ -part of Hodge-Tate weight $\frac{\partial \delta^{\mathrm{univ}}(x)}{\partial \sigma(x)}|_{x=1}$ of $\tilde{\delta}^{\mathrm{univ}}$ is not constant by lemma 3.19 of [Na10].

For any $1 \leq j_1 < j_2 < \dots < j_i \leq d$ such that $(j_1, j_2, \dots, j_i) \neq (1, 2, \dots, i)$, we put $\delta_{(j_1, \dots, j_i)} := (\prod_{k=1}^i \delta_{j_k} / \prod_{j=1}^i \delta_j)|_{\mathcal{O}_K^\times} : \mathcal{O}_K^\times \rightarrow E^\times$. We define a morphism of

functors $f_{(j_1, \dots, j_i)} : D_{V_x, \mathcal{T}_x} \rightarrow D_{\delta_{(j_1, \dots, j_i)}}$ as follows. For any $A \in \mathcal{C}_E$, we define a map $f_{(j_1, \dots, j_i)} : D_{V_x, \mathcal{T}_x}(A) \rightarrow D_{\delta_{(j_1, \dots, j_i)}}(A)$ by

$$f_{(j_1, \dots, j_i)}((V_A, \psi_A, \mathcal{T}_A)) := (\prod_{k=1}^i \delta_{j_k, A} / \prod_{j=1}^i \delta_{j, A})|_{\mathcal{O}_K^\times}$$

where the parameter of A -triangulation \mathcal{T}_A is $\{\delta_{j, A}\}_{j=1}^d$. We claim that this morphism is formally smooth, which proves the lemma because then $(\sum_{k=1}^i \frac{\partial \delta_{j_k, x}(x)}{\partial \sigma(x)}|_{x=1} - \sum_{j=1}^i \frac{\partial \delta_{j, x}(x)}{\partial \sigma(x)}|_{x=1})$ is equal to $f_{(j_1, \dots, j_i)}^*(\frac{\partial \delta^{\text{univ}}(x)}{\partial \sigma(x)}|_{x=1})$ where $f_{(j_1, \dots, j_i)}^* : R_{\delta_{(j_1, \dots, j_i)}} \hookrightarrow R_{V_x, \mathcal{T}_x}$ is the map induced by $f_{(j_1, \dots, j_i)}$ which is injection by the formal smoothness of $f_{(j_1, \dots, j_i)}$. We prove the claim. Let A be an object of \mathcal{C}_E and $I \subseteq A$ be an ideal of A such that $I^2 = 0$ and let $(V_{A/I}, \psi_{A/I}, \mathcal{T}_{A/I}) \in D_{V_x, \mathcal{T}_x}(A/I)$ and $\delta_A \in D_{(j_1, \dots, j_i)}(A)$ such that $(\prod_{k=1}^i \delta_{j_k, A/I} / \prod_{j=1}^i \delta_{j, A/I})|_{\mathcal{O}_K^\times} = \delta_A \pmod{I}$ where the parameter of A/I -triangulation of $\mathcal{T}_{A/I}$ is $\{\delta_{j, A/I}\}_{j=1}^d$. Because D_δ is formally smooth, we can take liftings $\delta_{i, A} : K^\times \rightarrow A^\times$ such that $\delta_{i, A} \pmod{I} = \delta_{i, A/I}$ for any $1 \leq i \leq d$ and $(\prod_{k=1}^i \delta_{j_k, A} / \prod_{j=1}^i \delta_{j, A})|_{\mathcal{O}_K^\times} = \delta_A$. Because we have $H^2(G_K, W(\delta_i/\delta_j)) = 0$ for any $i < j$ by the condition (ii) and proposition 2.9 of [Na10], the natural map $H^1(G_K, W(\delta_{1, A}/\delta_{2, A})) \rightarrow H^1(G_K, W(\delta_{1, A/I}/\delta_{2, A/I}))$ is surjection. Hence the sub (A/I) - B -pair $W_{2, A/I}$ of $W(V_{A/I})$ lifts to an A - B -pair $W_{2, A}$ which is an extension of $W(\delta_{1, A})$ by $W(\delta_{2, A})$. Repeating this procedure, we can take a trianguline A - B -pair (W_A, \mathcal{T}_A) which is a lift of $(W(V_{A/I}), \mathcal{T}_{A/I})$ and whose parameter is $\{\delta_{i, A}\}_{i=1}^d$. Moreover, there exists an A -representation V_A such that $W_A \xrightarrow{\sim} W(V_A)$ by proposition 1.5.6 of [Ke08]. This proves the formal smoothness of $f_{(j_1, \dots, j_i)}$. \square

Next, we prove that the local structure of $\mathcal{E}_{\overline{V}}$ at $z_{(V_x, \mathcal{T}_x)}$ can be described in terms of the trianguline deformation D_{V_x, \mathcal{T}_x} .

Theorem 3.9. *Let $z := z_{(V_x, \mathcal{T}_x)}$ be a point of $\mathcal{E}_{\overline{V}}$ satisfying all the conditions in proposition 3.6. If (V_x, \mathcal{T}_x) satisfies one of the following additional conditions (1) or (2),*

- (1) V_x is potentially crystalline and

$$\{a \in D_{\text{cris}}((\wedge^i V_x)(\prod_{j=1}^i \tilde{\delta}_j^{-1})) | \exists n \geq 1 \text{ such that } (\varphi^f - \prod_{j=1}^i \delta_j(\pi_K))^n a = 0\}$$

is a free of rank one $K_0 \otimes_{\mathbb{Q}_p} E$ -module for any $1 \leq i \leq d-1$,

- (2) *For any $1 \leq i \leq d-1$ and for any $1 \leq l_1 < l_2 < \dots < l_i \leq d$ such that $(l_1, \dots, l_i) \neq (1, 2, \dots, i)$, we have $(\sum_{j=1}^i k_{l_j, \sigma} - \sum_{j=1}^i k_{j, \sigma}) \notin \mathbb{Z}_{\leq 0}$ for any $\sigma \in \mathcal{P}$, where we define $k_{i, \sigma} := \frac{\partial \delta_i(x)}{\partial \sigma(x)}|_{x=1} \in E(z)$,*

then there exists a canonical isomorphism $\hat{\mathcal{O}}_{\mathcal{E}_{\overline{V}}, z} \xrightarrow{\sim} R_{V_x, \tau_x}$. In particular, $\mathcal{E}_{\overline{V}}$ is smooth of dimension $[K : \mathbb{Q}_P] \frac{d(d+1)}{2} + 1$ at z .

Proof. First, we claim that $D_0 := D_{\text{cris}}^+((\wedge^i V_x)(\prod_{j=1}^i \tilde{\delta}_j^{-1}))^{\varphi^f = \prod_{j=1}^i \delta_j(\pi_K)}$ is a sub E -filtered φ -module of $D_{\text{cris}}((\wedge^i V_x)(\prod_{j=1}^i \tilde{\delta}_j^{-1}))$ of rank one such that $\text{Fil}^1 K \otimes_{K_0} D_0 = 0$. Because we have $D_{\text{cris}}(W(\delta_{(\prod_{j=1}^i \delta_j(\pi_K))})) \subseteq D_0$ which is naturally defined from $\wedge^i(\mathcal{T})$ by lemma 3.8 of [Na10], it suffices to show that D_0 is at most rank one. This is trivial in the case of (1). In the case of (2), we assume that D_0 is not rank one, then if we denote by W' the cokernel of the natural injection $W(\delta_{(\prod_{j=1}^i \delta_j(\pi_K))}) \hookrightarrow W((\wedge^i V_x)(\prod_{j=1}^i \tilde{\delta}_j^{-1}))$ defined by $\wedge^i(\mathcal{T})$, the image of D_0 in $D_{\text{cris}}(W')$ is not zero. In particular, there exists a rank one E -filtered φ -submodule D of $D_{\text{cris}}(W')^{\varphi^f = \prod_{j=1}^i \delta_j(\pi_K)}$ such that $\text{Fil}^0 K \otimes_{K_0} D = K \otimes_{K_0} D$. This implies that W' has a Hodge-Tate weight $\{k_\sigma\}_{\sigma \in \mathcal{P}}$ ($k_\sigma \leq 0$), but, for any σ , any Hodge-Tate weights of W' are of the form $(\sum_{j=1}^i k_{l_j, \sigma} - \sum_{j=1}^i k_{j, \sigma})$ for some $\{l_1 < l_2 < \dots < l_i\} \neq \{1, 2, \dots, i\}$ which are not negative integer by the assumption, which is a contradiction. We finish the proof of the claim.

Next, we begin the proof of this theorem, as in the proof of Proposition 3.6, we may assume that $E(z) = E$. By the proof of proposition 3.6, we have already showed that there exists a natural local morphism

$$\hat{\mathcal{O}}_{\mathcal{E}_{\overline{V}}, z} \rightarrow R_z \xrightarrow{\sim} R_{V_x, \tau_x}.$$

We construct the inverse as follows. Let $\text{Spm}(R) \subseteq \mathcal{E}_{\overline{V}}$ be an affinoid neighborhood of z which is Y_i -small for any $1 \leq i \leq d-1$, this is possible because z is an E -rational point so we have $Y_i(z) \in E$ and we can take $\text{Spm}(R)$ such that $|\frac{Y_i}{Y_i(z)} - 1| < 1$ on $\text{Spm}(R)$. We take a sufficiently large $k \in \mathbb{Z}_{\geq 1}$ such that, for any $\sigma \in \mathcal{P}$ and $1 \leq i \leq d-1$, there exists a short exact sequence of Banach R -modules with property (Pr)

$$0 \rightarrow (B_{\max, K}^+ \hat{\otimes}_{K, \sigma} R)^{\varphi_K = \prod_{j=1}^i Y_j} \rightarrow B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} R \rightarrow U_{i, \sigma} \rightarrow 0$$

for a Banach R -module $U_{i, \sigma}$ with property (Pr). By proposition 3.2, we have isomorphisms

$$(B_{\max, K}^+ \hat{\otimes}_{K, \sigma} (N_i \otimes_{\mathcal{O}_{Z_0}} R))^{G_K, \varphi_K = \prod_{j=1}^i Y_j} \xrightarrow{\sim} (B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} (N_i \otimes_{\mathcal{O}_{Z_0}} R))^{G_K}$$

for any $\sigma \in \mathcal{P}$ and $1 \leq i \leq d-1$ and if we put $H_{i, \sigma}$ ($1 \leq i \leq d-1$, $\sigma \in \mathcal{P}$) the smallest ideal of R such that any G_K -equivariant R -linear map $h : N_i^\vee \rightarrow B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} R$ factors through $B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} H_{i, \sigma}$ and $H := \prod_{1 \leq i \leq d-1, \sigma \in \mathcal{P}} H_{i, \sigma}$, then $\text{Spm}(R) \setminus V(H)$ and $\text{Spm}(R) \setminus V(H_{i, \sigma})$ are scheme theoretically dense in $\text{Spm}(R)$. Under this situation, we denote by \tilde{T} the blow up of $\text{Spm}(R)$ along the ideal H and denote by $f : \tilde{T} \rightarrow \text{Spm}(R)$ the canonical projection. We show that, for any $\tilde{z} \in \tilde{T}$ such that $f(\tilde{z}) = z$, $N \otimes_{\mathcal{O}_Z} \mathcal{O}_{\tilde{T}, \tilde{z}} / \mathfrak{m}_{\tilde{z}}^n$ is a trianguline deformation of

$(V_x \otimes_E E(\tilde{z}), \mathcal{T}_x \otimes_E E(\tilde{z}))$ over $\mathcal{O}_{\tilde{T}, \tilde{z}}/\mathfrak{m}_{\tilde{z}}^n$ for any $n \in \mathbb{Z}_{\geq 1}$. To prove this claim, by the above claim, we first note that

$$D_{\text{cris}}^+(N_i \otimes_{\mathcal{O}_{Z_0}} E(\tilde{z}))^{\varphi^f = \prod_{j=1}^i Y_j(\tilde{z})} = D_{\text{cris}}^+((\wedge^i V_x)(\prod_{j=1}^i \tilde{\delta}_j^{-1}))^{\varphi^f = \prod_{j=1}^i \delta_j(\pi_K)} \otimes_E E(\tilde{z})$$

is a free $K_0 \otimes_{\mathbb{Q}_p} E(\tilde{z})$ -module of rank one and

$$\text{Fil}^1 D_{\text{dR}}(N_i \otimes_{\mathcal{O}_{Z_0}} E(\tilde{z})) \cap K \otimes_{K_0} D_{\text{cris}}^+(N_i \otimes_{\mathcal{O}_{Z_0}} E(\tilde{z}))^{\varphi^f = \prod_{j=1}^i Y_j(\tilde{z})} = 0.$$

By the definition of blow up, there exists a non zero divisor $h_{i,\sigma} \in \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}$ such that $H_{i,\sigma} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}} = h_{i,\sigma} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}$ for any $1 \leq i \leq d-1$ and $\sigma \in \mathcal{P}$. Then, by the definition of $H_{i,\sigma}$, for any $1 \leq i \leq d-1$ and $\sigma \in \mathcal{P}$, there exists a G_K -equivariant R -linear map $N_i^\vee \rightarrow (B_{\text{max}, K}^+ \hat{\otimes}_{K, \sigma} H_{i,\sigma})^{\varphi_K = \prod_{j=1}^i Y_j}$ such that the composite with

$$\begin{aligned} (B_{\text{max}, K}^+ \hat{\otimes}_{K, \sigma} H_{i,\sigma})^{\varphi_K = \prod_{j=1}^i Y_j} &\rightarrow (B_{\text{max}, K}^+ \hat{\otimes}_{K, \sigma} h_{i,\sigma} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}})^{\varphi_K = \prod_{j=1}^i Y_j} \\ &\xrightarrow{\sim} (B_{\text{max}, K}^+ \hat{\otimes}_{K, \sigma} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}})^{\varphi_K = \prod_{j=1}^i Y_j} \twoheadrightarrow (B_{\text{max}, K}^+ \otimes_{K, \sigma} E(\tilde{z}))^{\varphi_K = \prod_{j=1}^i \delta_j(\pi_K)} \end{aligned}$$

is non zero, where the isomorphism

$$(B_{\text{max}, K}^+ \hat{\otimes}_{K, \sigma} h_{i,\sigma} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}})^{\varphi_K = \prod_{j=1}^i Y_j} \xrightarrow{\sim} (B_{\text{max}, K}^+ \hat{\otimes}_{K, \sigma} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}})^{\varphi_K = \prod_{j=1}^i Y_j}$$

is given by $a \mapsto \frac{a}{h_{i,\sigma}}$. From these facts and from the condition (1), we can show (by induction on n) that $D_{\text{cris}}^+(N_i \otimes_{\mathcal{O}_{Z_0}} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}/\mathfrak{m}_{\tilde{z}}^n)^{\varphi_K = \prod_{j=1}^i Y_j}$ is a free $K_0 \otimes_{\mathbb{Q}_p} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}/\mathfrak{m}_{\tilde{z}}^n$ -module of rank one for any $1 \leq i \leq d-1$ and $n \in \mathbb{Z}_{\geq 1}$ and

$$K \otimes_{K_0} D_{\text{cris}}^+(N_i \otimes_{\mathcal{O}_{Z_0}} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}/\mathfrak{m}_{\tilde{z}}^n)^{\varphi^f = \prod_{j=1}^i Y_j} \cap \text{Fil}^1 D_{\text{dR}}(N_i \otimes_{\mathcal{O}_{Z_0}} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}/\mathfrak{m}_{\tilde{z}}^n) = 0.$$

By proposition 3.10 below, then $N \otimes_{\mathcal{O}_Z} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}/\mathfrak{m}_{\tilde{z}}^n$ is a trianguline deformation of $(V_x \otimes_E E(\tilde{z}), \mathcal{T}_x \otimes_E E(\tilde{z}))$ over $\hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}/\mathfrak{m}_{\tilde{z}}^n$ whose parameter is $\{\delta_i^{\text{univ}} \pmod{\mathfrak{m}_{\tilde{z}}^n}\}_{i=1}^d$, so the natural map $R_{V_x} \rightarrow \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}$ factors through $R_{V_x, \mathcal{T}_x} \rightarrow \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}$. This shows that the natural map $R_{V_x} \rightarrow \hat{\mathcal{O}}_{\mathcal{E}_{\tilde{V}, z}}$ sends the kernel of the quotient map $R_{V_x} \rightarrow R_{V_x, \mathcal{T}_x}$ to the kernel of the natural map

$$g : \hat{\mathcal{O}}_{\mathcal{E}_{\tilde{V}, z}} \rightarrow \prod_{\tilde{z} \in \tilde{T}, f(\tilde{z})=z} \hat{\mathcal{O}}_{\tilde{T}, \tilde{z}}.$$

Because g is injection by lemma 10.7 of [Ki03] and by proposition 3.2 (2), the natural map $R_{V_x} \rightarrow \hat{\mathcal{O}}_{\mathcal{E}_{\tilde{V}, z}}$ also factors thorough $R_{V_x, \mathcal{T}_x} \rightarrow \hat{\mathcal{O}}_{\mathcal{E}_{\tilde{V}, z}}$. We can check that this gives the desired inverse map. The last statement of the theorem follows from proposition 2.39 of [Na10]. \square

The following proposition is a generalization of theorem 2.5.6 of [BeCh09] for any p -adic field case.

Proposition 3.10. *Let V be a split trianguline E -representation of rank d with a triangulation $\mathcal{T} : 0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_d := W(V)$ with the parameter $\{\delta_i\}_{i=1}^d$. We assume that (V, \mathcal{T}) satisfies one of conditions (1) or (2) of Theorem 3.9. Let $A \in \mathcal{C}_E$ and V_A be a deformation of V over A and, for any $1 \leq i \leq d$, $\delta_{i,A} : K^\times \rightarrow A^\times$ be a continuous homomorphism which is a lift of δ_i satisfying the following conditions.*

- (1) *For any $1 \leq i \leq d-1$, $D_{\text{cris}}^+((\wedge^i V_A)(\prod_{j=1}^i \tilde{\delta}_{j,A}^{-1}))^{\varphi^f = \prod_{j=1}^i \delta_{j,A}(\pi_K)}$ is a free $K_0 \otimes_{\mathbb{Q}_p} A$ -module of rank one.*
- (2) *For any $1 \leq i \leq d-1$, the natural base change map*

$$D_{\text{cris}}^+((\wedge^i V_A)(\prod_{j=1}^i \tilde{\delta}_{j,A}^{-1}))^{\varphi^f = \prod_{j=1}^i \delta_{j,A}(\pi_K)} \otimes_A E \rightarrow D_{\text{cris}}^+((\wedge^i V)(\prod_{j=1}^i \tilde{\delta}_j))_{\varphi^f = \prod_{j=1}^i \delta_j(\pi_K)}$$

is isomorphism.

Then, V_A has an A -triangulation \mathcal{T}_A such that (V_A, \mathcal{T}_A) is a deformation of (V, \mathcal{T}) and the parameter of \mathcal{T}_A is $\{\delta_{i,A}\}_{i=1}^d$.

Proof. The proof is of course essentially same as that of [BeCh09], but we give the proof here for the convenience of readers. We put $L_i := (\wedge^i V)(\prod_{j=1}^i \tilde{\delta}_j^{-1})$ and $L_{i,A} := (\wedge^i V_A)(\prod_{j=1}^i \tilde{\delta}_{j,A}^{-1})$ and $\lambda_i := \delta_i(\pi_K)$ and $\lambda_{i,A} := \delta_{i,A}(\pi_K)$ for any $1 \leq i \leq d-1$. By the claim in the proof of the above Theorem, we have

$$K \otimes_{K_0} D_{\text{cris}}^+(L_i)^{\varphi^f = \prod_{j=1}^i \lambda_j} \cap \text{Fil}^1 D_{\text{dR}}(L_i) = 0,$$

then by induction on the length of A , we can also show that

$$K \otimes_{K_0} D_{\text{cris}}^+(L_{i,A})^{\varphi^f = \prod_{j=1}^i \lambda_{j,A}} \cap \text{Fil}^1 D_{\text{dR}}(L_{i,A}) = 0.$$

From this and the condition (1), $D_{\text{cris}}^+(L_{i,A})^{\varphi^f = \prod_{j=1}^i \lambda_{j,A}}$ is an A -filtered φ -module, hence by the condition (2) and lemma 2.22 of [Na10], there exists an A -saturated inclusion

$$h_i : W(\prod_{j=1}^i \delta_{\lambda_{j,A}}) \hookrightarrow W(L_{i,A})$$

such that $D_{\text{cris}}(h_i)$ corresponds to the canonical injection

$$D_{\text{cris}}^+(L_{i,A})^{\varphi^f = \prod_{j=1}^i \lambda_{j,A}} \hookrightarrow D_{\text{cris}}(L_{i,A}).$$

Twisting this injection by $\prod_{j=1}^i \tilde{\delta}_{j,A}$, we obtain an A -saturated injection

$$h'_i : W(\prod_{j=1}^i \delta_{i,A}) \hookrightarrow W(\wedge^i V_A)$$

such that $h'_i(\bmod \mathfrak{m}_A)$ is equal to the canonical injection

$$W\left(\prod_{j=1}^i \delta_j\right) \hookrightarrow W(\wedge^i V)$$

which is naturally induced by $\wedge^i \mathcal{T}$. Using these facts, we show by induction that there exists an A -saturated sub A - B -pair $W_{i,A}$ of $W(V_A)$ such that $W_{j,A}$ is a sub A -saturated A - B -pair of $W_{j+1,A}$ and $W_{j+1,A}/W_{j,A} \xrightarrow{\sim} W(\delta_{j+1,A})$ for any $1 \leq j \leq i-1$ and that the image of inclusion $W_{i,A} \otimes_A E \hookrightarrow W(V_A) \otimes_A E = W(V)$ is equal to W_i . First, we take $W_{1,A}$ as the image of the inclusion $h'_1 : W(\delta_{1,A}) \hookrightarrow W(V_A)$. We assume that we can take $W_{1,A} \subseteq W_{2,A} \subseteq \cdots \subseteq W_{i-1,A} \subseteq W(V_A)$ satisfying the above conditions. If we denote by $W'_{i,A}$ (resp. W'_i) the cokernel of the inclusion $W_{i-1,A} \hookrightarrow W(V_A)$ (resp. $W_{i-1} \hookrightarrow W(V)$), taking \wedge^i of $W(V_A)$ (resp. $W(V)$), we obtain a following short exact sequence of A - B -pairs,

$$0 \rightarrow (\wedge^{i-1} W_{i-1,A}) \otimes W'_{i,A} \rightarrow W(\wedge^i V_A) \rightarrow W''_{i,A} \rightarrow 0$$

for an A - B -pair $W''_{i,A}$ (resp. an E - B -pair W''_i). Under this situation, we claim that the map $W(\prod_{j=1}^i \delta_{j,A}) \rightarrow W''_{i,A}$ (which is defined as the composite of h'_i with the canonical projection $W(\wedge^i V_A) \rightarrow W''_{i,A}$) is zero. By dévissage, it suffices to show that the natural map $W(\prod_{j=1}^i \delta_j) \rightarrow W''_i$ is zero. We first prove this claim in the case of (1) of Theorem 3.9. Because $D_{\text{cris}}(W(\delta_{(\prod_{j=1}^i \lambda_j)}))^{\varphi^f = \prod_{j=1}^i \lambda_j} \neq 0$, then it suffices to show that $D_{\text{cris}}(W''_i(\prod_{j=1}^i \tilde{\delta}_j^{-1}))^{\varphi^f = \prod_{j=1}^i \lambda_j} = 0$, which follows from the condition (1) of Theorem 3.9. Next, we prove the claim in the case of (2) of Theorem 3.9. If the map $W(\prod_{j=1}^i \delta_j) \rightarrow W''_i$ is non zero, we also have an injection $W(\delta_{(\prod_{j=1}^i \lambda_j)}) \hookrightarrow W''_i(\prod_{j=1}^i \tilde{\delta}_j^{-1})$, but this injection implies that $W''_i(\prod_{j=1}^i \tilde{\delta}_j^{-1})$ has a Hodge-Tate weight $\{k_\sigma\}_{\sigma \in \mathcal{P}}$ ($k_\sigma \in \mathbb{Z}_{\leq 0}$), which contradicts the condition (2).

By this claim, the map $W(\prod_{j=1}^i \delta_{j,A}) \hookrightarrow W(\wedge^i V_A)$ factors through an A -saturated injection $W(\prod_{j=1}^i \delta_{j,A}) \hookrightarrow (\wedge^{i-1} W_{i-1,A}) \otimes W'_{i,A}$. Because $\wedge^{i-1} W_{i-1,A} \xrightarrow{\sim} W(\prod_{j=1}^{i-1} \delta_{j,A})$, we obtain an A -saturated injection $W(\delta_{i,A}) \hookrightarrow W'_{i,A}$. If we define $W_{i,A} (\subseteq W(V_A))$ as the inverse image of $W(\delta_{i,A}) (\subseteq W'_{i,A})$ by the natural projection $W(V_A) \rightarrow W'_{i,A}$, $W_{i,A}$ satisfies all the desired properties, which proves the proposition. \square

We define a morphism $f_0 : Z \rightarrow \mathcal{W}^{\times d}$ by $f(z) := (\delta_1|_{\mathcal{O}_K^\times}, \dots, \delta_{d-1}|_{\mathcal{O}_K^\times}, \delta_d|_{\mathcal{O}_K^\times})$ for any point $z := (x, \delta_1, \dots, \delta_{d-1}) \in Z$, where if $x \in X$ corresponds to an $E(x)$ -representation V_x then we define $\delta_d := \det(V_x) / (\prod_{j=1}^{d-1} \delta_j)$. We denote by

$$f : \mathcal{E}_{\overline{V}} \rightarrow \mathcal{W}^{\times d}$$

the composition of f_0 with the canonical immersion $\mathcal{E}_{\overline{V}} \hookrightarrow Z$.

Proposition 3.11. *Let $z := z_{(V_x, \mathcal{T}_x)}$ be a point of $\mathcal{E}_{\overline{V}}$ satisfying all the conditions in theorem 3.9. Then, f is smooth at z .*

Proof. By theorem 3.9, we have an isomorphism $\widehat{\mathcal{O}}_{\mathcal{E}_{\overline{V}}, z} \xrightarrow{\sim} R_{V_x, \mathcal{T}_x}$. Moreover, we have the following natural isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{W}^{\times d}, f(z)} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{W}, (\delta_1|_{\mathcal{O}_K^\times})} \hat{\otimes}_{E(z)} \cdots \hat{\otimes}_{E(z)} \widehat{\mathcal{O}}_{\mathcal{W}, (\delta_d|_{\mathcal{O}_K^\times})} \xrightarrow{\sim} R_{(\delta_1|_{\mathcal{O}_K^\times})} \hat{\otimes}_{E(z)} \cdots \hat{\otimes}_{E(z)} R_{(\delta_d|_{\mathcal{O}_K^\times})}$$

where $R_{(\delta_i|_{\mathcal{O}_K^\times})}$ is the universal deformation ring defined in the proof of lemma 3.8. If we identify

$$D_{V_x, \mathcal{T}_x}(A) = \mathrm{Spf}(R_{V_x, \mathcal{T}_x})(A)$$

and

$$D_{(\delta_1|_{\mathcal{O}_K^\times})}(A) \times \cdots \times D_{(\delta_d|_{\mathcal{O}_K^\times})}(A) = \mathrm{Spf}(R_{(\delta_1|_{\mathcal{O}_K^\times})} \hat{\otimes}_{E(z)} \cdots \hat{\otimes}_{E(z)} R_{(\delta_d|_{\mathcal{O}_K^\times})})(A)$$

for any $A \in \mathcal{C}_{E(z)}$, the local morphism at z induced by f is equal to the morphism

$$f_z : \mathrm{Spf}(R_{V_x, \mathcal{T}_x}) \rightarrow \mathrm{Spm}(R_{(\delta_1|_{\mathcal{O}_K^\times})} \hat{\otimes}_{E(z)} \cdots \hat{\otimes}_{E(z)} R_{(\delta_d|_{\mathcal{O}_K^\times})})$$

whose A -valued points for any $A \in \mathcal{C}_{E(z)}$ are given by

$$D_{V_x, \mathcal{T}_x}(A) \rightarrow D_{(\delta_1|_{\mathcal{O}_K^\times})}(A) \times \cdots \times D_{(\delta_d|_{\mathcal{O}_K^\times})}(A) : (V_A, \mathcal{T}_A, \psi_A) \mapsto (\delta_{1,A}|_{\mathcal{O}_K^\times}, \dots, \delta_{d,A}|_{\mathcal{O}_K^\times})$$

where $\{\delta_{i,A}\}_{i=1}^d$ is the parameter of \mathcal{T}_A . We claim that this morphism is formally smooth. Let $A \in \mathcal{C}_{E(z)}$ and $I \subseteq A$ be an ideal such that $I\mathfrak{m}_A = 0$. Let $(V_{A/I}, \psi_{A/I}, \mathcal{T}_{A/I}) \in D_{V_x, \mathcal{T}_x}(A/I)$ and $(\delta'_{1,A}, \delta'_{2,A}, \dots, \delta'_{d,A}) \in D_{(\delta_1|_{\mathcal{O}_K^\times})}(A) \times \cdots \times D_{(\delta_d|_{\mathcal{O}_K^\times})}(A)$ such that $f_{z(V_x, \mathcal{T}_x)}((V_{A/I}, \psi_{A/I}, \mathcal{T}_{A/I})) = (\delta'_{1,A}, \dots, \delta'_{d,A}) \pmod{I}$. Take a lift $\{\delta_{i,A}\}_{i=1}^d$ ($\delta_{i,A} : K^\times \rightarrow A^\times$) of the parameter $\{\delta_{i,A/I}\}_{i=1}^d$ of $(V_{A/I}, \mathcal{T}_{A/I})$ such that $\delta_{i,A}|_{\mathcal{O}_K^\times} = \delta'_{i,A}$. Because $H^2(G_K, W(\delta_i/\delta_j)) = 0$ for any $i < j$ by proposition 2.9 of [Na10], so we have $H^2(G_K, W_i(\delta_{i+1}^{-1})) = 0$ for any $1 \leq i \leq d-1$. From this, we obtain a surjection $H^1(G_K, W(\delta_{1,A}/\delta_{2,A})) \rightarrow H^1(G_K, W(\delta_{1,A/I}/\delta_{2,A/I}))$, hence we can take a lift $[W_{2,A}] \in H^1(G_K, W(\delta_{1,A}/\delta_{2,A}))$ of $[W_{2,A/I}] \in H^1(G_K, W(\delta_{1,A/I}/\delta_{2,A/I}))$, where $\mathcal{T}_{A/I} : 0 \subseteq W_{1,A/I} \subseteq W_{2,A/I} \subseteq \cdots \subseteq W_{d,A/I} = W(V_{A/I})$. Then, the natural map $H^1(G_K, W_{2,A}(\delta_{3,A}^{-1})) \rightarrow H^1(G_K, W_{2,A/I}(\delta_{3,A/I}^{-1}))$ is also surjection, hence we can take a lift $[W_{3,A}] \in H^1(G_K, W_{2,A}(\delta_{3,A}^{-1}))$ of $[W_{3,A/I}] \in H^1(G_K, W_{2,A/I}(\delta_{3,A/I}^{-1}))$. Repeating this procedure inductively, we obtain a trianguline A - B -pair W_A with a triangulation $\mathcal{T}_A : 0 \subseteq W_{1,A} \subseteq W_{2,A} \subseteq \cdots \subseteq W_{d,A} = W_A$ with the parameter $\{\delta_{i,A}\}_{i=1}^d$ such that $[W_{i,A}] \in H^1(G_K, W_{i-1,A}(\delta_{i,A}^{-1}))$ is a lift of $[W_{i,A/I}] \in H^1(G_K, W_{i-1,A/I}(\delta_{i,A/I}^{-1}))$. This shows that f_z is formally smooth. Because this property is preserved by base change of the base field E , f is smooth at z by proposition 2.9 of [BLR95]. □

Let (V_x, \mathcal{T}_x) be as in the above proposition such that $z_{(V_x, \mathcal{T}_x)}$ is an E -rational point of $\mathcal{E}_{\overline{V}}$. We denote by $\{k_{1,\sigma}, k_{2,\sigma}, \dots, k_{d,\sigma}\}_{\sigma \in \mathcal{P}}$ the Hodge-Tate weight of V_x such that $k_{1,\sigma} \geq k_{2,\sigma} \geq \dots \geq k_{d,\sigma}$ for any $\sigma \in \mathcal{P}$. We take an affinoid open neighborhood $U = \text{Spm}(R) \subseteq \mathcal{E}_{\overline{V}}$ of $z_{(V_x, \mathcal{T}_x)}$ which is Y_i -small for any $1 \leq i \leq d-1$ such that, for any $1 \leq i \leq d$, $v_i := v_p(\delta'_i(\pi_K))$ is constant for any $z = (V', \delta'_1, \dots, \delta'_{d-1}) \in U$. Take a sufficiently large k such that

- (1) For any $1 \leq i \leq d-1$ and $\sigma \in \mathcal{P}$, there exists a following short exact sequence of Banach R -modules with property (Pr),

$$0 \rightarrow (B_{\max, K}^+ \hat{\otimes}_{K, \sigma} R)^{\varphi_K = \prod_{j=1}^i Y_j} \rightarrow B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} R \rightarrow U_{i, \sigma} \rightarrow 0$$

for a Banach R -module $U_{i, \sigma}$ with property (Pr).

- (2) $k > \frac{\max\{2, (d-1)^2\}}{f} \max_{1 \leq i \leq d} \{|v_i|\}$.

We fix k which satisfies the above conditions. We define a subset $\mathcal{W}_k^{\times d}$ of $\mathcal{W}^{\times d}$ by

$$\mathcal{W}_k^{\times d} := \{(\prod_{\sigma \in \mathcal{P}} \sigma^{k'_{1, \sigma}}, \prod_{\sigma \in \mathcal{P}} \sigma^{k'_{2, \sigma}}, \dots, \prod_{\sigma \in \mathcal{P}} \sigma^{k'_{d, \sigma}}) \in \mathcal{W}^{\times d} \mid k'_{i, \sigma} - k'_{i+1, \sigma} > k \text{ for any } 1 \leq i \leq d-1, \sigma \in \mathcal{P}\}.$$

The following proposition is a crucial proposition to prove that the subset consisting of crystalline representations is (locally) Zariski dense in $\mathcal{E}_{\overline{V}}$.

Proposition 3.12. *Under the above situation, let $z := (V', \delta'_1, \dots, \delta'_{d-1})$ be a point of $U \cap f^{-1}(\mathcal{W}_k^{\times d})$. Then, V' is a crystalline and split trianguline $E(z)$ -representation with a triangulation \mathcal{T}' whose parameter is $\{\delta'_i\}_{i=1}^d$ (where $\delta'_d := \det(V') / \prod_{i=1}^{d-1} \delta'_i$), i.e. $z = z_{(V', \mathcal{T}')}$.*

Proof. First, by the definition of U and by proposition 3.2, we have an isomorphism

$$(B_{\max, K}^+ \hat{\otimes}_{K, \sigma} (N_i \otimes_{\mathcal{O}_{Z_0}} R))^{\varphi_K = \prod_{j=1}^i Y_j} \xrightarrow{\sim} (B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} (N_i \otimes_{\mathcal{O}_{Z_0}} R))^{\varphi_K}$$

for any $1 \leq i \leq d-1$ and $\sigma \in \mathcal{P}$. Because $f(z) \in \mathcal{W}_k^{\times d}$, we have $Q_{i, \sigma}(j)(z) \neq 0$ for any $\sigma \in \mathcal{P}$ and $1 \leq i \leq d-1$ and $-k \leq j \leq 0$. Hence, by corollary 2.6 of [Ki03], the natural base change map is an isomorphism

$$(B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \hat{\otimes}_{K, \sigma} (N_i \otimes_{\mathcal{O}_{Z_0}} R))^{\varphi_K} \otimes_R E(z) \xrightarrow{\sim} (B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \otimes_{K, \sigma} (\wedge^i V')) \left(\prod_{j=1}^i \tilde{\delta}_j'^{-1} \right)^{G_K}$$

and this is one-dimensional over $E(z)$ for any σ and i . The natural map

$$(B_{\max, K}^+ \otimes_{K, \sigma} E(z))^{\varphi_K = \prod_{j=1}^i \delta'_j(\pi_K)} \hookrightarrow B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \otimes_{K, \sigma} E(z)$$

is an injection by the definition of U . From these facts, the natural map

$$(B_{\max, K}^+ \hat{\otimes}_{K, \sigma} (\wedge^i V')) \left(\prod_{j=1}^i \tilde{\delta}_j'^{-1} \right)^{G_K, \varphi_K = \prod_{j=1}^i \delta'_j(\pi_K)} \xrightarrow{\sim} (B_{\text{dR}}^+ / t^k B_{\text{dR}}^+ \otimes_{K, \sigma} (\wedge^i V')) \left(\prod_{j=1}^i \tilde{\delta}_j'^{-1} \right)^{G_K}$$

is an isomorphism for any $\sigma \in \mathcal{P}$ and $1 \leq i \leq d-1$. From this isomorphism and because $f(z) \in \mathcal{W}_k^{\times d}$, we can check that $D_i := D_{\text{cris}}^+((\wedge^i V')(\prod_{j=1}^i \tilde{\delta}_j'^{-1}))^{\varphi^f = \prod_{j=1}^i \delta_j'(\pi_K)}$ is a sub rank one $E(z)$ -filtered φ -module of $D_{\text{cris}}((\wedge^i V')(\prod_{j=1}^i \tilde{\delta}_j'^{-1}))$ such that $\text{Fil}^0(K \otimes_{K_0} D_i) = K \otimes_{K_0} D_i$ and $\text{Fil}^1(K \otimes_{K_0} D_i) = 0$ for any $1 \leq i \leq d-1$. Hence, by lemma 2.21 of [Na10], we obtain a saturated injection

$$W(\delta_{(\prod_{j=1}^i \delta_j'(\pi_K))}) \hookrightarrow W((\wedge^i V')(\prod_{j=1}^i \tilde{\delta}_j'^{-1}))$$

and, twisting this by $\prod_{j=1}^i \tilde{\delta}_j'$, we obtain a following saturated injection

$$W(\prod_{j=1}^i \delta_j') \hookrightarrow W(\wedge^i V')$$

for any $1 \leq i \leq d-1$. Then, by proposition 3.13 below, V' is split trianguline $E(z)$ -representation with a triangulation $\mathcal{T}' : 0 \subseteq W'_1 \subseteq W'_2 \subseteq \cdots \subseteq W'_d = W(V')$ whose parameter is equal to $\{\delta'_i\}_{i=1}^d$. To finish the proof, it suffices to show that W'_i is crystalline for any $1 \leq i \leq d$ by induction on i . By lemma 3.14 below, it suffices to check that the parameter $\{\delta'_i\}_{i=1}^d$ satisfies the conditions (1) and (2) in this lemma. For (1), it is trivial by the definition of $\mathcal{W}_k^{\times d}$. For (2), if $\delta'_i/\delta'_j = \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma} |N_{K/\mathbb{Q}_p}|_p$ for some $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\geq 1}$ and for some $1 \leq i < j \leq d$, then $k_\sigma = k_{i,\sigma} - k_{j,\sigma} \geq k + 1$. Hence the slope of $W(\delta'_i/\delta'_j)$ is $\frac{1}{[K:\mathbb{Q}_p]}(\sum_{\sigma \in \mathcal{P}} k_\sigma) - 1 \geq k$. On the other hands, the slope of $W(\delta'_i/\delta'_j)$ can be computed by $\frac{1}{f}(v_p(\delta'_i(\pi_K)) - v_p(\delta'_j(\pi_K))) < k$, this is a contradiction. Hence $\{\delta'_i\}_{i=1}^d$ satisfies (1), (2) of lemma 3.14, hence V' is crystalline. \square

Proposition 3.13. *Let $z := (V, \delta_1, \dots, \delta_{d-1}) \in Z_0$ be a point which satisfies the following conditions (1) and (2), then V is a split trianguline $E(z)$ -representation with a triangulation \mathcal{T} whose parameter is $\{\delta_i\}_{i=1}^d$, i.e. $z = z_{(V, \mathcal{T})}$.*

(1) *For any $1 \leq i \leq d-1$, there exists a saturated injection*

$$W(\prod_{j=1}^i \delta_j) \hookrightarrow W(\wedge^i V).$$

(2) *One of the following conditions holds,*

- (i) *If we put $k_{i,\sigma} := \frac{\partial \delta_i(x)}{\partial \sigma(x)}|_{x=1} \in E(z)$, then for any $1 \leq i \leq d-1$ and for any $1 \leq l_1 < l_2 < \cdots < l_i \leq d$ such that $\{1, 2, \dots, i\} \neq \{l_1, \dots, l_i\}$ and for any $\sigma \in \mathcal{P}$, $(\sum_{j=1}^i k_{j,\sigma} - \sum_{j=1}^i k_{l_j,\sigma}) \notin \mathbb{Z}_{\geq 0}$.*
- (ii) *All $k_{i,\sigma}$ are integers and, if we put $v_0 := \max_{1 \leq i \leq d} \{v_p(\delta_i(\pi_K))\}$, $k_{i,\sigma} - k_{i+1,\sigma} > \frac{(d-1)^2}{f} v_0$ for any $1 \leq i \leq d-1$ and $\sigma \in \mathcal{P}$.*

Proof. First, by the condition (1) for $i = 1$, we have a saturated injection $W(\delta_1) \hookrightarrow W(V)$. We put $W_1 \subseteq W(V)$ the image of this injection. By induction on i , we show that we can take a filtration $0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{i-1} \subseteq W_i \subseteq W(V)$ such that W_i is a rank i $E(z)$ - B -pair which is saturated in $W(V)$ and $W_{j+1}/W_j \xrightarrow{\sim} W(\delta_{j+1})$ for any $1 \leq j \leq i-1$ and $\wedge^i W_i \subseteq W(\wedge^i V)$ is equal to the image of the given injection $W(\prod_{j=1}^i \delta_j) \hookrightarrow W(\wedge^i V)$ in (1). We assume that we can take $0 \subseteq W_1 \subseteq \cdots \subseteq W_{i-1} \subseteq W(V)$ satisfying all the above conditions. If we put W' the cokernel of the injection $W_{i-1} \subseteq W(V)$ and if we take i -th wedge product, we obtain a following short exact sequence of $E(z)$ - B -pairs,

$$0 \rightarrow W\left(\prod_{j=1}^{i-1} \delta_j\right) \otimes W' \rightarrow W(\wedge^i V) \rightarrow W'' \rightarrow 0$$

because $\wedge^{i-1} W_{i-1} \xrightarrow{\sim} W(\prod_{j=1}^{i-1} \delta_j)$, where W'' is a successive extension of $\wedge^j W_{i-1} \otimes \wedge^{i-j} W'$ for $0 \leq j \leq i-2$. By the condition (1), we have an injection $W(\prod_{j=1}^i \delta_j) \hookrightarrow W(\wedge^i V)$, we denote $\iota : W(\prod_{j=1}^i \delta_j) \rightarrow W''$ the composition of this injection with the canonical surjection $W(\wedge^i V) \twoheadrightarrow W''$. We claim that $\iota : W(\prod_{j=1}^i \delta_j) \rightarrow W''$ is a zero map under the condition (2).

First, we prove this claim under the condition (i) of (2). In this case, if ι is not zero, then this is injection because $W(\prod_{j=1}^i \delta_j)$ is rank one. Then, by proposition 2.14 of [Na09], the saturation of the image of ι is isomorphic to $W(\prod_{j=1}^i \delta_j \prod_{\sigma \in \mathcal{P}} \sigma^{-k_\sigma})$ for some $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\geq 0}$. This implies that W'' has a Hodge-Tate weight $\{\sum_{j=1}^i k_{j,\sigma} - k_\sigma\}_{\sigma \in \mathcal{P}}$. But, because $z \in Z_0$ and the Hodge-Tate weights of W_{i-1} is $\{k_{1,\sigma}, \dots, k_{i-1,\sigma}\}_{\sigma \in \mathcal{P}}$, any σ -part of Hodge-Tate weights of W'' are equal to $\sum_{l=1}^i k_{j_l,\sigma}$ for some $1 \leq j_1 < \cdots < j_i \leq d$ such that $\{1, 2, \dots, i\} \neq \{j_1, \dots, j_i\}$, this contradicts to the condition (i), hence ι must be zero.

Next, we prove the claim under the condition (ii). We assume that ι is not zero. Because W'' is a successive extension of $\wedge^j W_{i-1} \otimes \wedge^{i-j} W'$ for $0 \leq j \leq i-2$, we obtain a saturated injection $W(\prod_{j=1}^i \delta_j \prod_{\sigma \in \mathcal{P}} \sigma^{-k_\sigma}) \hookrightarrow \wedge^j W_{i-1} \otimes \wedge^{i-j} W'$ for some $0 \leq j \leq i-2$ and for some $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\geq 0}$. Because, any σ -part of Hodge-Tate weight of W'' is equal to $\sum_{l=1}^i k_{j_l,\sigma}$ for some $1 \leq j_1 < \cdots < j_i \leq d$ such that $\{j_1, \dots, j_i\} \neq \{1, 2, \dots, i\}$, we have $k_\sigma = \sum_{j=1}^i k_{j,\sigma} - \sum_{l=1}^i k_{j_l,\sigma} > \frac{(d-1)^2}{f} v_0$ by the condition (ii). Because the slope of $W(\prod_{j=1}^i \delta_j \prod_{\sigma \in \mathcal{P}} \sigma^{-k_\sigma})$ is equal to

$$\frac{1}{f} \left(\sum_{j=1}^i v_p(\delta_j(\pi_K)) \right) - \frac{1}{[K : \mathbb{Q}_p]} \left(\sum_{\sigma \in \mathcal{P}} k_\sigma \right) < \frac{i}{f} v_0 - \frac{(d-1)^2}{f} v_0 = \frac{i - (d-1)^2}{f} v_0,$$

the smallest slope (which we denote by s'') of $\wedge^j W_{i-1} \otimes \wedge^{i-j} W'$ satisfies $s'' < \frac{i - (d-1)^2}{f} v_0$. On the other hands, because we have an injection $W(\prod_{j=1}^{i-1} \delta_j) \otimes W' \hookrightarrow$

$W(\wedge^i V)$, the smallest slope s' of W' satisfies $s' \geq -\frac{1}{f}(\sum_{j=1}^{i-1} v_p(\pi_K)) \geq -\frac{(i-1)}{f}v_0$ by corollary 1.6.9 of [Ke08]. Because all the slopes of W_{i-1} are positive or zero by corollary 1.6.9 of [Ke08], then the smallest slope s'' of $\wedge^j W_{i-1} \otimes \wedge^{i-j} W'$ satisfies that $s'' \geq \min\{(is', 2s')\} \geq -\frac{(i-1)i}{f}v_0$ by remark 1.7.2 of [Ke08]. Hence, we obtain an inequality $-\frac{(i-1)i}{f}v_0 \leq s'' < \frac{i-(d-1)^2}{f}v_0$, which implies that $(d-1)^2 < i^2$, this is a contradiction, hence ι must be zero. We finish to prove the claim in both cases.

This claim implies that the given injection $W(\prod_{j=1}^i \delta_j) \hookrightarrow W(\wedge^i V)$ factors through a saturated injection $W(\prod_{j=1}^i \delta_j) \hookrightarrow W(\prod_{j=1}^{i-1} \delta_j) \otimes W'$. Twisting this injection by $\prod_{j=1}^{i-1} \delta_j^{-1}$, we obtain a saturated injection $W(\delta_i) \hookrightarrow W'$. If we denote $W_i(\subseteq W(V))$ the inverse image of $W(\delta_i)(\subseteq W')$ by the canonical surjection $W(V) \twoheadrightarrow W'$, we obtain a following short exact sequence,

$$0 \rightarrow W_{i-1} \rightarrow W_i \rightarrow W(\delta_i) \rightarrow 0$$

and we can check that W_i satisfies the desired properties. By induction, we finish to prove this proposition. \square

Lemma 3.14. *Let W be a split trianguline E - B -pair of rank d with a triangulation $\mathcal{T} : 0 \subseteq W_1 \subseteq \cdots \subseteq W_{d-1} \subseteq W_d = W$ with the parameter $\{\delta_i\}_{i=1}^d$. If $\{\delta_i\}_{i=1}^d$ satisfies the following conditions (1) and (2), then W is crystalline.*

- (1) *For any $1 \leq i \leq d$, $\delta_i|_{\mathcal{O}_K^\times} = \prod_{\sigma \in \mathcal{P}} \sigma^{k_{i,\sigma}}$ for some $\{k_{i,\sigma}\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}$ such that $k_{1,\sigma} > k_{2,\sigma} > \cdots > k_{d,\sigma}$ for any $\sigma \in \mathcal{P}$.*
- (2) *For any $1 \leq i < j \leq d$, $\delta_i/\delta_j \neq \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma} |N_{K/\mathbb{Q}_p}|_p$ for any $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\geq 1}$.*

Proof. We prove this lemma by induction on the rank of W . If W is rank one, the condition (1) implies that $W = W(\delta_1)$ is crystalline. We assume that W is of rank d and W_{d-1} is crystalline. Then we claim that the natural injection $H_f^1(G_K, W_{d-1}(\delta_d^{-1})) \hookrightarrow H^1(G_K, W_{d-1}(\delta_d^{-1}))$ is bijection, which proves that W is crystalline (where for any B -pair W , $H_f^1(G_K, W)$ is Bloch-Kato cohomology of W defined in definition 2.4 of [Na09]). We prove this claim by computing the dimensions of both E -vector spaces. First we have $H^2(G_K, W(\delta_i/\delta_d)) = 0$ for any $1 \leq i \leq d-1$ by the condition (2) and proposition 2.9 of [Na10]. Because $W_{d-1}(\delta_d^{-1})$ is a successive extension of $W(\delta_i/\delta_d)$, we also have $H^2(G_K, W_{d-1}(\delta_d^{-1})) = 0$. Then, we have

$$\dim_E H^1(G_K, W_{d-1}(\delta_d^{-1})) = [K : \mathbb{Q}_p](d-1) + \dim_E H^0(G_K, W_{d-1}(\delta_d^{-1}))$$

by Euler-Poincaré formula (theorem 2.8 of [Na10]). On the other hands, because $W_{d-1}(\delta_d^{-1})$ is crystalline, we have

$$\begin{aligned} \dim_E H_f^1(G_K, W_{d-1}(\delta_d^{-1})) &= \dim_E D_{\text{dR}}(W_{d-1}(\delta_d^{-1})) / \text{Fil}^0 D_{\text{dR}}(W_{d-1}(\delta_d^{-1})) \\ &\quad + \dim_E H^0(G_K, W_{d-1}(\delta_d^{-1})) \end{aligned}$$

by proposition 2.7 of [Na09]. Because $W_{d-1}(\delta_d^{-1})$ is a successive extension of $W(\delta_i/\delta_d)$, the condition (1) implies that $\mathrm{Fil}^0 D_{\mathrm{dR}}(W_{d-1}(\delta_d^{-1})) = 0$, hence we have

$$\begin{aligned} \dim_E H_f^1(G_K, W_{d-1}(\delta_d^{-1})) &= \dim_E D_{\mathrm{dR}}(W_{d-1}(\delta_d^{-1})) + \dim_E H^0(G_K, W_{d-1}(\delta_d^{-1})) \\ &= [K : \mathbb{Q}_p](d-1) + \dim_E H^0(G_K, W_{d-1}(\delta_d^{-1})) \\ &= \dim_E H^1(G_K, W_{d-1}(\delta_d^{-1})). \end{aligned}$$

We finish to prove the claim, hence we finish to prove the lemma. \square

We define two subsets $\mathfrak{X}_{\overline{V}, \mathrm{reg-cris}}$ and $\mathfrak{X}_{\overline{V}, b}$ of $\mathfrak{X}_{\overline{V}}$ by

$$\mathfrak{X}_{\overline{V}, \mathrm{reg-cris}} := \{x = [V_x] \in \mathfrak{X}_{\overline{V}} \mid \text{Hodge-Tate weight } \{k_{i,\sigma}\}_{1 \leq i \leq d, \sigma \in \mathcal{P}} \text{ of } V_x \text{ satisfies } k_{i,\sigma} \neq k_{j,\sigma} \text{ for any } i \neq j, \sigma \in \mathcal{P}\}$$

$$\mathfrak{X}_{\overline{V}, b} := \{x = [V_x] \in \mathfrak{X}_{\overline{V}} \mid V_x \otimes_{E(x)} E' \text{ is benign for a finite extension } E' \text{ of } E(x) \text{ and } \mathcal{T}_\tau \text{ satisfies the condition (1) of theorem 3.9 for any } \tau \in \mathfrak{S}_d\}$$

From the above proposition, we can prove the following theorem, which states Zariski density of crystalline points in $\mathcal{E}_{\overline{V}}$.

Theorem 3.15. *Let $z_{(V_x, \mathcal{T}_x)} \in \mathcal{E}_{\overline{V}}$ be an E -rational point satisfying all the conditions in theorem 3.9. Then, for any admissible open neighborhood $U \subseteq \mathcal{E}_{\overline{V}}$ of $z_{(V_x, \mathcal{T}_x)}$, there exists a smaller admissible open neighborhood $U' \subseteq U$ of $z_{(V_x, \mathcal{T}_x)}$ such that the subset defined by*

$$U'_{\mathrm{cris}} := \{z = ([V], \delta_1, \dots, \delta_{d-1}) \in U' \mid [V] \in \mathfrak{X}_{\overline{V}, \mathrm{reg-cris}}\}$$

is Zariski dense in U' .

Proof. If we use proposition 3.11 and proposition 3.12, the proof of this theorem is same as that of lemma 4.7 of [Na10]. \square

4. ZARISKI DENSITY OF CRYSTALLINE REPRESENTATIONS FOR ANY p -ADIC FIELD

Lemma 4.1. *Let $x = [V_x] \in \mathfrak{X}_{\overline{V}, \mathrm{reg-cris}}$ be a point. Then, for any admissible open neighborhood $U \subseteq \mathfrak{X}_{\overline{V}}$ of x , $U \cap \mathfrak{X}_{\overline{V}, b}$ is not empty.*

Proof. The proof is a generalization of the proof of lemma 4.12 of [Na10]. Of course, we may assume that $E(x) = E$. The Hodge-Tate weight $\tau := \{k_{i,\sigma}\}_{1 \leq i \leq d, \sigma \in \mathcal{P}}$ of V_x satisfies that $k_{1,\sigma} > k_{2,\sigma} > \dots > k_{d,\sigma}$ for any $\sigma \in \mathcal{P}$. Then the subset $\mathfrak{X}_{\overline{V}, \mathrm{cris}}^\tau$ of $\mathfrak{X}_{\overline{V}}$ consisting of the points corresponding to crystalline representations with Hodge-Tate weight $\{k_{i,\sigma}\}_{1 \leq i \leq d, \sigma \in \mathcal{P}}$ forms a Zariski closed subspace of $\mathfrak{X}_{\overline{V}}$ corresponding to a quotient $R_{\overline{V}, \mathrm{cris}}^\tau$ of $R_{\overline{V}}$ by Corollary 2.7.7 of [Ki08]. We consider the universal framed deformation ring $R_{\overline{V}}^\square$ of (\overline{V}, β) , where β is a fixed \mathbb{F} -base of \overline{V} . Then, in the same way as $R_{\overline{V}, \mathrm{cris}}^\tau$, we obtain a quotient $R_{\overline{V}, \mathrm{cris}}^{\square, \tau}$ of $R_{\overline{V}}^\square$ and we have a map

$R_{\overline{V}, \text{cris}}^\tau \rightarrow R_{\overline{V}, \text{cris}}^{\square, \tau}$ which is naturally induced from the map $R_{\overline{V}} \rightarrow R_{\overline{V}}^{\square}$ corresponding to the forgetting map $D_{\overline{V}}^{\square}(A) \rightarrow D_{\overline{V}}(A) : (V_A, \psi_A, \tilde{\beta}) \mapsto (V_A, \psi_A)$ ($A \in \mathcal{C}_O$) where $\tilde{\beta}$ is an A -base of V_A which is a lift of β . Therefore, if we put $\mathfrak{X}_{\overline{V}, \text{cris}}^{\square, \tau}$ the rigid analytic space associated to $R_{\overline{V}, \text{cris}}^{\square, \tau}$, it suffices to show the following lemma. \square

Lemma 4.2. *Let x be a point of $\mathfrak{X}_{\overline{V}, \text{cris}}^{\square, \tau}$ and let U be an admissible open neighborhood of x , then there exists a point z of U whose corresponding representation is benign and satisfies the condition (1) of theorem 3.9.*

Proof. We remark that, by the proof of Theorem 3.3.8 of [Ki08], we have an isomorphism $\hat{\mathcal{O}}_{\mathfrak{X}_{\overline{V}, \text{cris}}^{\square, \tau}, y} \xrightarrow{\sim} R_{V_y}^{\square, \text{cris}}$ for any point $y \in \mathfrak{X}_{\overline{V}, \text{cris}}^{\square, \tau}$. Then, by Corollary 6.3.3 of [Be-Co08] and by Corollary 3.19 of [Ch09a], there exists an admissible affinoid open neighborhood $U = \text{Spm}(R)$ of x in $\mathfrak{X}_{\overline{V}, \text{cris}}^{\square, \tau}$, such that $D_{\text{cris}}(V_R) := ((R \hat{\otimes}_{\mathbb{Q}_p} B_{\text{cris}}) \otimes_R V_R)^{G_K}$ is a finite free $K_0 \otimes_{\mathbb{Q}_p} R$ -module of rank d and $D_{\text{dR}}(V_R) \xrightarrow{\sim} K \otimes_{K_0} D_{\text{cris}}(V_R)$, where V_R is the restriction to U of the universal deformation of \overline{V} . For any $\sigma' \in \text{Gal}(K_0/\mathbb{Q}_p)$, we denote by $D_{\sigma'}$ the σ' -component of $D_{\text{cris}}(V_R)$. We denote by $T^d + a_{d-1}T^{d-1} + \cdots + a_1T + a_0 := \det_R(\text{Id}_{D_{\sigma'}} - \varphi^f|_{D_{\sigma'}}) \in R[T]$ the characteristic polynomial of relative Frobenius on $D_{\sigma'}$, which does not depend on the choice of σ' . Let $\Delta \in R$ be the discriminant of this polynomial. Then, we claim that Δ is a non zero divisor of R , i.e. the subset $U_{\Delta} \subseteq U$ consisting of points z such that $D_{\text{cris}}(V_z)$ has d distinct relative Frobenius eigenvalues is scheme theoretically dense in U . For proving this claim, it suffices to show that $\Delta \neq 0$ in $\hat{\mathcal{O}}_{\mathfrak{X}(\overline{\rho})_{\text{cris}}^{\square, \tau}, z} \xrightarrow{\sim} R_{V_z}^{\square, \text{cris}}$ for any $z \in U$ because $R_{V_z}^{\square, \text{cris}}$ is domain. But it is easy to see that $D_{\text{cris}}(V_z)$ can be deformed to $E(z)[\varepsilon]$ such that with d distinct relative Frobenius eigenvalues, hence $\Delta \neq 0$ in $R_{V_z}^{\square, \text{cris}}$. In the same way, we can show that the subset $U'' \subseteq U$ consisting of points z such that $D_{\text{cris}}(V_z)$ has relative Frobenius eigenvalues $\{\alpha_i\}_{1 \leq i \leq d}$ satisfying $\alpha_i \neq p^{\pm f} \alpha_j$ ($i \neq j$) is also scheme theoretically dense in U , hence their intersection $U_{\Delta} \cap U''$ is also scheme theoretically dense in U . Next, we take an element $z \in U_{\Delta} \cap U'' \subseteq U$, then by extending scalar, we may assume that $D_{\text{cris}}(V_z) \xrightarrow{\sim} \oplus_{i=1}^d K_0 \otimes_{\mathbb{Q}_p} E e_{i,z}$ such that $\varphi^f(e_{i,z}) = \alpha_{i,z} e_{i,z}$ for some $\alpha_{i,z} \in E^{\times}$ ($1 \leq i \leq d$) such that $\alpha_{i,z} \neq \alpha_{j,z} p^{\pm f}$ for any $i \neq j$. Because $\mathcal{O}_{\mathfrak{X}(\overline{\rho})_{\text{cris}}^{\square, \tau}, z}$ is Henselian by Theorem 2.1.5 of [Berk93], if we take a sufficiently small affinoid open neighborhood $U' = \text{Spm}(R')$ of z in $U_{\Delta} \cap U''$, then we have $D_{\text{cris}}(V_R) \otimes_R R' \xrightarrow{\sim} \oplus_{i=1}^d K_0 \otimes_{\mathbb{Q}_p} R' e_i$ such that $K_0 \otimes_{\mathbb{Q}_p} R' e_i$ is φ -stable and $\varphi^f(e_i) = \tilde{\alpha}_i e_i$ for some $\tilde{\alpha}_i \in R'^{\times}$ for $1 \leq i \leq d$ satisfying that $\tilde{\alpha}_i - \tilde{\alpha}_j, \tilde{\alpha}_i - p^{\pm f} \tilde{\alpha}_j \in R'^{\times}$ for any $i \neq j$. Then, for sufficiently small U' and for any $\sigma \in \mathcal{P}$, if we decompose $D_{\text{dR}}(V_R) \otimes_R R'$ into σ -component by $D_{\text{dR}}(V_R) \otimes_R R' \xrightarrow{\sim} D_{\text{dR}}(V_{R'}) = \oplus_{\sigma \in \mathcal{P}} D_{\sigma}$, then σ -component D_{σ} of $D_{\text{dR}}(V_{R'})$ is equipped with a filtration by finite free R' -modules $\text{Fil}^i D_{\sigma}$ such that $\text{Fil}^i D_{\sigma} \otimes_{R'} B \xrightarrow{\sim} \text{Fil}^i D_{\text{dR}}(V_{R'} \otimes_{R'} B)_{\sigma}$ for any local R' -algebra which

is finite over E by lemma 2.6.1 and by the proof of corollary 2.6.2 of [Ki08]. From these facts, we obtain two R' -basis $\{e_{i,\sigma}\}_{i=1}^d$ and $\{f_{i,\sigma}\}_{i=1}^d$ of D_σ , where $\{e_{i,\sigma}\}_{i=1}^d$ is the base naturally induced from the base $\{e_i\}_{i=1}^d$ of $D_{\text{cris}}(V_{R'})$ and $\{f_{i,\sigma}\}_{i=1}^d$ satisfies that $\text{Fil}^{-k_{i,\sigma}} D_\sigma = R' f_{i,\sigma} \oplus R' f_{i+1,\sigma} \oplus \cdots \oplus R' f_{d,\sigma}$ for any $1 \leq i \leq d$. For any $\sigma \in \mathcal{P}$, we define $A_\sigma := (a_{i,j,\sigma})_{i,j}$ by $f_{i,\sigma} := \sum_{j=1}^d a_{i,j,\sigma} e_{j,\sigma}$. We denote by $a \in R$ the product of all k -th minor determinants of A_σ for all $1 \leq k \leq d-1$ and $\sigma \in \mathcal{P}$. By the definition of benign representations, for any $z \in \text{Spm}(R')$, it is easy to see that V_z is benign if and only if $a(z) \neq 0$ in $E(z)$. Therefore, to prove the lemma, it suffices to show that $\text{Spm}(R')_a$ and $\text{Spm}(R')_{(\prod_{j=1}^i \tilde{\alpha}_{k_j} - \prod_{j=1}^i \tilde{\alpha}_{l_j})}$ (for any $1 \leq i \leq d-1$ and $\{k_1 < k_2 < \cdots < k_i\} \neq \{l_1 < \cdots < l_i\} \subseteq \{1, 2, \dots, d\}$) are scheme theoretically dense in $\text{Spm}(R')$, i.e. it suffices to show that a and $(\prod_{j=1}^i \tilde{\alpha}_{k_j} - \prod_{j=1}^i \tilde{\alpha}_{l_j})$ are non zero divisors of R' . Because we have an isomorphism $\hat{R}'_{\mathfrak{m}_z} \xrightarrow{\sim} R'_{V_z}^{\square, \text{cris}}$ for any $z \in \text{Spm}(R')$ and $R'_{V_z}^{\text{cris}}$ is domain, it suffices to show that $a, (\prod_{j=1}^i \tilde{\alpha}_{k_j} - \prod_{j=1}^i \tilde{\alpha}_{l_j}) \in R'_{V_z}^{\square, \text{cris}}$ are non-zero for any $z \in \text{Spm}(R')$. Finally, this claim can be proved by constructing explicitly lifts of filtered φ -module $D_{E(z)[\varepsilon]}$ of $D_{\text{cris}}(V_z)$ over $E(z)[\varepsilon]$ such that $a \neq 0$ or $(\prod_{j=1}^i \tilde{\alpha}_{k_j} - \prod_{j=1}^i \tilde{\alpha}_{l_j}) \neq 0$ in $E(z)[\varepsilon]$. \square

For a rigid analytic space Y over E and for a point $y \in Y$, we denote by

$$t_{Y,y} := \text{Hom}_{E(y)}(\mathfrak{m}_y/\mathfrak{m}_y^2, E(y))$$

the tangent space at y , where \mathfrak{m}_y is the maximal ideal of $\mathcal{O}_{Y,y}$.

We denote by $\overline{\mathfrak{X}}_{\overline{V}, \text{reg-cris}}$ the Zariski closure of $\mathfrak{X}_{\overline{V}, \text{reg-cris}}$ in $\mathfrak{X}_{\overline{V}}$.

The following theorems are the main theorems of this paper.

Theorem 4.3. $\overline{\mathfrak{X}}_{\overline{V}, \text{reg-cris}}$ is a union of irreducible components of $\mathfrak{X}_{\overline{V}}^{\text{red}}$.

Proof. Because any irreducible components of $\mathfrak{X}_{\overline{V}}$ are of dimension at most $[K : \mathbb{Q}_p]d^2 + 1$, so it suffices to show that any irreducible components of $\overline{\mathfrak{X}}_{\overline{V}, \text{reg-cris}}$ have $[K : \mathbb{Q}_p]d^2 + 1$ dimension. Let Z' be an irreducible component of $\overline{\mathfrak{X}}_{\overline{V}, \text{reg-cris}}$. Because the singular locus $Z'_{\text{sing}} \subseteq Z'$ is a proper Zariski closed set of Z' , there exists a point $x \in \mathfrak{X}_{\overline{V}, \text{b}} \cap Z'$ such that Z' (and $\mathfrak{X}_{\overline{V}}$) is smooth at x by lemma 4.1. By the definition of benign representation and by proposition 3.6, the point $z_{(V_x, \mathcal{T}_\tau)} \in Z_0$ corresponding to the pair (V_x, \mathcal{T}_τ) is contained in $\mathcal{E}_{\overline{V}}$ for any $\tau \in \mathfrak{S}_d$. We denote by Y_τ the irreducible component of $p^{-1}(\overline{\mathfrak{X}}_{\overline{V}, \text{reg-cris}})$ containing $z_{(V_x, \mathcal{T}_\tau)}$ for $\tau \in \mathfrak{S}_d$. Because the natural morphism $p|_{Y_\tau} : Y_\tau \rightarrow \mathfrak{X}_{\overline{V}}$ factors through Z' for any $\tau \in \mathfrak{S}_d$, we obtain a map

$$t_{\mathcal{E}_{\overline{V}}, z_{(V_x, \mathcal{T}_\tau)}} = t_{Y_\tau, z_{(V_x, \mathcal{T}_\tau)}} \rightarrow t_{Z', x} \hookrightarrow t_{\mathfrak{X}_{\overline{V}}, x}$$

for any $\tau \in \mathfrak{S}_d$, where the first equality follows from theorem 3.15. Hence, we obtain a map

$$\bigoplus_{\tau \in \mathfrak{S}_d} t_{\mathcal{E}_{\overline{V}}, z_{(V_x, \mathcal{T}_\tau)}} \rightarrow t_{Z', x} \hookrightarrow t_{\mathfrak{X}_{\overline{V}}, x}.$$

By theorem 2.61 of [Na10] and theorem 3.9, this map is surjective, hence we obtain an equality

$$t_{Z',x} = t_{\mathfrak{X}_{\overline{V}},x}.$$

Because x is smooth at Z' , then Z' has dimension $[K : \mathbb{Q}_p]d^2 + 1$. This proves the theorem. \square

Let $\omega : G_K \rightarrow \mathbb{F}^\times$ be the mod p cyclotomic character and let $\text{ad}(\overline{V}) := \text{End}_{\mathbb{F}}(\overline{V})$ and $\text{ad}(\overline{V})^0 := \text{ad}(\overline{V})^{\text{trace}=0}$.

Theorem 4.4. *We assume that \overline{V} satisfies the following conditions,*

- (0) $\text{End}_{\mathbb{F}[G_K]}(\overline{V}) = \mathbb{F}$.
- (1) $\mathfrak{X}_{\overline{V},\text{reg-cris}}$ is non empty.
- (2) $H^0(G_K, \text{ad}(\overline{V})^0(\omega)) = 0$.
- (3) $\zeta_p \notin K^\times$ or $p \nmid d$.

Then, we have an equality $\overline{\mathfrak{X}}_{\overline{V},\text{reg-cris}} = \mathfrak{X}_{\overline{V}}$.

Proof. First, we prove the theorem when $\zeta_p \notin K^\times$. It suffices to show that $\mathfrak{X}_{\overline{V}}$ is irreducible by theorem 4.3. This claim follows from the fact that $H^2(G_K, \text{ad}(\overline{V})) = 0$, which follows from the condition (2) and the fact that $H^0(G_K, \mathbb{F}(\omega)) = 0$ when $\zeta_p \notin K^\times$.

Next, we prove the theorem when $p \nmid d$. Let P be the sub group of \mathcal{O}_K^\times consisting of all p -th power roots of unity. Let p^n be the order of P , we take $\zeta_{p^n} \in \mathcal{O}_K^\times$ a primitive p^n -th roots of unity. For any $1 \leq i \leq p^n - 1$, we define a subfunctor D_i of $D_{\overline{V}}$ by

$$D_i(A) := \{[(V_A, \psi_A)] \in D_{\overline{V}}(A) \mid \det(V_A)(\text{rec}_K(\zeta_{p^n})) = \iota_A(\zeta_{p^n})^i\}$$

for any $A \in \mathcal{C}_{\mathcal{O}}$, where $\iota_A : \mathcal{O} \rightarrow A$ is the morphism which gives the \mathcal{O} -algebra structure on A . In the same way as in the proof of theorem 4.16 of [Na10], we can prove that, under the condition (2), D_i is representable by a quotient R_i of $R_{\overline{V}}$ which is formally smooth over \mathcal{O} . Moreover, if we denote by \mathfrak{X}_i the rigid analytic space associated to R_i , then we have an equality as rigid space

$$\mathfrak{X}_{\overline{V}} = \coprod_{0 \leq i \leq p^n - 1} \mathfrak{X}_i$$

and \mathfrak{X}_i is irreducible. By the condition (1) and by theorem 4.3, there exists i such that $\mathfrak{X}_i \subseteq \overline{\mathfrak{X}}_{\overline{V},\text{reg-cris}}$. Because $\chi_{\text{LT}}^{p^f-1} \equiv 1 \pmod{\pi_K}$ and $\chi_{\text{LT}}(\zeta_{p^n}) = \zeta_{p^n}$, twisting by $\chi_{\text{LT}}^{(p^f-1)m}$ ($m \in \mathbb{Z}$) induces an isomorphism $\mathfrak{X}_i \xrightarrow{\sim} \mathfrak{X}_{i+(p^f-1)dm \pmod{p^n}}$. Because χ_{LT} is crystalline, this isom implies that $\mathfrak{X}_{i+(p^f-1)dm \pmod{p^n}} \subseteq \overline{\mathfrak{X}}_{\overline{V},\text{reg-cris}}$. Because $p \nmid d$, any $j \pmod{p^n}$ can be written by $i + (p^f - 1)dm \pmod{p^n}$ for some $m \in \mathbb{Z}$, hence we have an equality $\overline{\mathfrak{X}}_{\overline{V},\text{reg-cris}} = \mathfrak{X}_{\overline{V}}$. \square

Finally, when \overline{V} is absolutely irreducible, we can prove the following theorem, which is a generalization of theorem A of [Ch10]

Theorem 4.5. *We assume that*

- (1) \overline{V} is absolutely irreducible,
- (2) $\zeta_p \in K$ or $\overline{V} \not\cong \overline{V}(\omega)$,

then we have an equality $\overline{\mathfrak{X}}_{\overline{V}, \text{reg-cris}} = \mathfrak{X}_{\overline{V}}$.

Proof. First, we note that the condition (2) is equivalent to the condition that $H^0(G_K, \text{ad}(\overline{V})^0(\omega)) = 0$ under the assumption (1). We use the notation used in the proof of theorem 4.4. Let K_d be the unramified extension of K such that $[K_d : K] = d$. By extending E , we assume that $\text{Hom}_{\mathbb{Q}_p\text{-alg}}(K_d, E) = \text{Hom}_{\mathbb{Q}_p\text{-alg}}(K_d, \overline{K})$. Let $\chi_d : G_{K_d}^{\text{ab}} \rightarrow \mathbb{F}_{q^d}^\times \xrightarrow{\sim} (\mathcal{O}_{K_d}/\pi_K \mathcal{O}_{K_d})^\times \hookrightarrow \mathbb{F}$ be the fundamental character of degree d , i.e. the character defined by $\chi_d(\text{rec}_{K_d}(a)) := \bar{a}$ ($a \in \mathcal{O}_{K_d}^\times$), $\chi_d(\text{rec}_{K_d}(\pi_K)) = 1$. Then it is known that $\overline{V} \xrightarrow{\sim} \text{Ind}_{G_{K_d}}^{G_K}(\chi_d^i) \otimes_{\mathbb{F}} \mathbb{F}(\eta)$ for some $i \in \mathbb{Z}$ and $\eta : G_K^{\text{ab}} \rightarrow \mathbb{F}^\times$. Because any η has a crystalline lift, we may assume that $\overline{V} \xrightarrow{\sim} \text{Ind}_{G_{K_d}}^{G_K}(\chi_d^i)$. By theorem 4.3 and by the proof of theorem 4.4, it suffices to show that $\mathfrak{X}_i \cap \mathfrak{X}_{\overline{V}, \text{reg-cris}}$ (resp. $\mathfrak{X}_{\overline{V}, \text{reg-cris}}$) is non empty for any $0 \leq i \leq p^n - 1$ when $\zeta_p \in K^\times$ (resp. $\zeta_p \notin K^\times$). To prove this claim, let $\chi_{d, \text{LT}} : G_{K_d}^{\text{ab}} \rightarrow K_d^\times$ be the Lubin-Tate character of K_d associated to $\pi_K \in K \subseteq K_d$ and let τ be a generator of $\text{Gal}(K_d/K)$ and, for any $\sigma \in \mathcal{P}$, let $\tilde{\sigma} : K_d \rightarrow E$ be a \mathbb{Q}_p -algebra homomorphism such that $\tilde{\sigma}|_K = \sigma$. Then, there exists $\{a_{\sigma, i}\}_{\sigma \in \mathcal{P}, 0 \leq i \leq d-1} \in \prod_{\sigma \in \mathcal{P}, 0 \leq i \leq d-1} \mathbb{Z}$ such that $a_{\sigma, i} \neq a_{\sigma, j}$ for any $i \neq j$ and that

$$\chi_d^i \equiv \prod_{\sigma \in \mathcal{P}, 0 \leq i \leq d-1} (\tilde{\sigma} \tau^i(\chi_{d, \text{LT}}))^{a_{\sigma, i}} \pmod{\pi_E}$$

and

$$\prod_{\sigma \in \mathcal{P}, 0 \leq i \leq d-1} (\tilde{\sigma} \tau^i(\chi_{d, \text{LT}}))^{a_{\sigma, i}} \equiv \prod_{\sigma \in \mathcal{P}, 1 \leq i \leq d-1} (\tilde{\sigma} \tau^i(\chi_{d, \text{LT}}))^{a'_{\sigma, i}} \pmod{\pi_E}$$

if $a_{\sigma, i} \equiv a'_{\sigma, i} \pmod{(p^{fd} - 1)}$ for any (σ, i) . Then,

$$V_{\{a_{\sigma, i}\}_{\sigma, i}} := \text{Ind}_{G_{K_d}}^{G_K} \left(\prod_{\sigma \in \mathcal{P}, 0 \leq i \leq d-1} (\tilde{\sigma} \tau^i(\chi_{d, \text{LT}}))^{a_{\sigma, i}} \right)$$

is a lift of \overline{V} which is a crystalline representation whose σ -part of Hodge-Tate weight is $\{a_{\sigma, i}\}_{0 \leq i \leq d-1}$, i.e. this is an element of $\mathfrak{X}_{\overline{V}, \text{reg-cris}}$, which proves the claim when $\zeta_p \notin K$. Moreover, because

$$\chi_{\{a_{\sigma, i}\}_{\sigma, i}} := \det(\text{Ind}_{G_{K_d}}^{G_K} \left(\prod_{\sigma \in \mathcal{P}, 0 \leq i \leq d-1} (\tilde{\sigma} \tau^i(\chi_{d, \text{LT}}))^{a_{\sigma, i}} \right))$$

is crystalline character whose σ -part of Hodge-Tate weight is $b_\sigma := \sum_{i=0}^{d-1} a_{\sigma, i}$, we have $\chi_{\{a_{\sigma, i}\}_{\sigma, i}} \circ \text{rec}_K|_{\mathcal{O}_K^\times} = \prod_{\sigma \in \mathcal{P}} \sigma^{b_\sigma}$. Therefore, if we denote by $\sigma(\zeta_{p^n}) := \zeta_{p^n}^{c_\sigma}$

($c_\sigma \in (\mathbb{Z}/p^n)^\times$) for any $\sigma \in \mathcal{P}$, we have $[V_{\{a_{\sigma,i}\}_{\sigma,i}}] \in \mathfrak{X}_{i_0}$, where $i_0 \equiv \sum_{\sigma \in \mathcal{P}} b_\sigma c_\sigma \pmod{p^n}$. We take a $\sigma \in \mathcal{P}$ and $m \in \mathbb{Z}$, we define $\{a'_{\sigma,i}\}_{\sigma,i}$ such that $a'_{\sigma,0} := a_{\sigma,0} + m(p^{fd} - 1)$ and $a'_{\sigma,i} = a_{\sigma,i}$ otherwise. The above arguments implies that $[V_{\{a'_{\sigma,i}\}_{\sigma,i}}] \in \mathfrak{X}_{i_m} \cap \mathfrak{X}_{\overline{V}, \text{reg-cris}}$, where $i_m = i_0 + c_\sigma(p^{fd} - 1)m$ for all but finitely many $m \in \mathbb{Z}$. Because $\{i_m\}_{m \in \mathbb{Z}}$ runs thorough all \mathbb{Z}/p^n , this implies that $\mathfrak{X}_i \cap \mathfrak{X}_{\overline{V}, \text{reg-cris}}$ is non empty for any $0 \leq i \leq p^n - 1$, which proves the claim. \square

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